THE

INDUCTION MOTOR

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THE THEORY, DESIGN, AND APPLICATION
OF ALTERNATING-CURRENT MACHINES
INCLUDING FRACTIONAL H.P. MOTORS

BY

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Preface to the Second Edition

In the new edition of this book opportunity has been taken to enlarge the scope of its contents and to bring it up to date. Most of the book has been entirely rewritten. An attempt has been made to cover the theory, design, and applications of the motor. It may appear to some that emphasis has been laid too heavily on design and design principles, but it is my belief that the machine is best understood, in all its forms, when design principles are thoroughly mastered. Much additional material has been introduced in connection with single-phase machines, selsyns, the three-phase series and shunt commutator motors, a new theory of the single-phase motor, and also the application of symmetrical components to conditions of unbalance. Fractional-horse-power motors have received fairly full treatment and typical designs are worked out for single- and three-phase types.

In preparing and writing this second edition, I am greatly indebted to the works of the late B. A. Behrend, of Boston—the great pioneer in this field; to the work of Prof. Waldo Lyon; to Messrs. Wagner and Evans; to Dr. Liwschitz-Garik; and to the American Institute of Electrical Engineers. I also wish to express grateful thanks to The Westinghouse Electric Corporation for photos of fractional h.p. machines; to The B.T.H. Co., of Rugby, for photos, curves, and data on control and selsyns; to The English Electric Co., of Stafford, for excellent photos of machines, windings, and regulators; and to Messrs. Clarke Chapman for photos of machines and windings, and also for drawings. I hope the book will have a wide appeal to students and engineers.

H. VICKERS

February, 1949

Preface to the First Edition

THIS book is intended to introduce the student to the theory and design of induction motors.

An attempt has been made to deal with the latest developments in speed and power-factor control, and to incorporate most of the theory connected with the motor and its applications.

While the scope of the book is large, it is hoped that no section of the work is lacking in thoroughness. An endeavour has been made to place the design of induction motors on a firmer scientific basis than it has rested upon heretofore.

In this work I have largely drawn on my own experience, but I am deeply conscious of my great debt to various writers.

In the course of the work many technical journals, and especially the *Proceedings of the American Institute of Electrical Engineers*, have been consulted and, in addition, use has been made of the works of Steinmetz.

Due acknowledgment has been made, throughout the book, of the sources from which the information has been drawn.

It is hoped that the book will make a large appeal to engineers and students of the various technical colleges and universities.

H. VICKERS

VANCOUVER
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INSETS

Introductory

It will be interesting to the student of electrical engineering to have some account of the fascinating history of the induction motor. The discovery by Gambey, the instrument-maker of Paris, that a compass needle, when disturbed and set oscillating, comes to rest more quickly when it is in the vicinity of copper, than when wood is near it, was made in 1824. At that time also Barlow and Marsh, at Woolwich, had observed the effect on a magnetic needle of rotating it near a sphere of iron. Arago published, in 1824, an account of an experiment with a compass needle within rings of different materials. In this experiment he pushed the needle aside to about 45° and counted the number of oscillations made by the needle before the swing decreased to 10°. With a ring of wood the number of oscillations was 145; with a copper ring 66; and with a stout copper ring only 33. In 1825 he suspended a compass needle over a rotating copper disc and found that, by turning the disc slowly, the needle is deviated out of the magnetic meridian. By rotating the disc fast enough he found that continuous rotation of the needle could be produced. The brilliant discovery by Faraday, in 1831, of electro-magnetic induction provided the solution to the question of the origin of the forces present in the above experiments of Gambey, Barlow and Marsh, and Arago. Faraday showed that the rotation of the Arago disc was due to induced currents, set up in the disc by relative motion of disc and compass needle. From 1831 to 1879 this valuable discovery produced no further results. In June, 1879, Mr. Walter Baily read a paper, before the Physical Society of London, on "A Mode of Producing Arago's Rotations." Baily used a fixed electromagnet with four magnet cores joined to a yoke.

The four magnet cores were about 4 in. long and each was wound with about 150 turns of insulated copper wire of 2·5 mm diameter. The coils were connected two and two in series, similar to two independent horse-shoe magnets and were set diagonally across one to another.

The two circuits were connected separately to a revolving commutator, built up of a simple arrangement of springs and contact strips mounted on a piece of wood, with a wire handle by which it was turned. By rotation, the currents from two batteries were caused to be reversed alternately in the two circuits, and this gave rise to the following changes in polarity of the four poles.

N	O	N	N	O	N	S	N	S	O
↘		↓		↙		←		↖	
O	S	S	S	S	O	S	N	O	N

In this rotating magnetic field a copper disc was suspended. He stated: "The rotation of the disc is due to that of the magnetic field in which it is suspended, and we should expect that, if a similar motion of the field could be produced by any other means the result would be a similar motion of the disc." He also suggested that if a whole circle of poles were arranged under the disc, successively excited in opposite pairs, the series of impulses all tend to make the disc revolve in one direction around the axis, and added: "In one extreme case, when the number of electromagnets is infinite, we have the case of a uniform rotation of the magnetic field, such as we obtain by rotating permanent magnets." It is clear that Mr. Baily had grasped the fundamental principle of action of the induction motor, and the motor he exhibited before the Physical Society, in 1879, was the first induction motor, but it needed later important discoveries of methods for producing the revolving field by means of alternating currents to make it the useful machine that it is to-day. The next discovery was made by Marcel Deprez in 1883.

Deprez fed alternating current to a coil, which produced an alternating or oscillating field along the OX axis. He supplied another coil, whose magnetic axis made an angle of 90° with the OX axis, with alternating current, whose phase difference was 90° in time from the current in the first coil, and showed that a revolving field of constant amplitude could be produced. The frequency of the two currents was the same. He also showed that if the two currents were of equal period, but not of equal amplitude, an elliptically rotating field was produced. The number of turns in each coil was the same.

Professor Ferraris arrived at the same conclusions as Baily and Deprez in 1885, and apparently without knowing of the work of either. His paper on "Electrodynamic Rotations Produced by Means of Alternating Currents" was published in 1888. He suggested the method of obtaining currents, differing in phase by nearly 90°, by inserting a resistance in one winding and inductance in the other, thus making the ratio of $\frac{\text{reactance}}{\text{resistance}}$ small in one winding and large in the other. This method, it may be noted, is largely used for starting up single-phase motors.

Then followed the great work of Nikola Tesla between 1887 and 1891. His researches placed the induction motor on a sound foundation. His patents were sold to the Westinghouse Co. of America, whose pioneer efforts in this field must be recognized. In that period, however, the only a.c. supply circuits were single phase, and the frequencies were 133 and 125 c/s. These supply circuits were obviously unsuitable for the development of the motor.

In 1891 the Electrotechnical Exhibition at Frankfort was held, and three-phase transmission of power was demonstrated. Two turbine-driven three-phase generators were installed at Lauffen,

generating 1400 A at 55 V. The frequency was 40 c/s. Three-phase transformers were installed, at each end of the line, to raise the voltage to 8000 V at Lauffen, and to reduce it to 65 V at Frankfort. The distance of transmission was 110 miles. This bold experiment demonstrated the feasibility of three-phase transmission of energy. The load consisted of a 100 h.p. three-phase motor and also lamps. Several German firms exhibited different types of three-phase induction motors at the exhibition. One three-phase, 3 b.h.p. motor had the three-phase supply brought into the rotor by three slip-rings, the secondary circuit, being the stator winding, consisted of a closed-circuit winding. Several motors, built by the Oerlikon Co., and designed by the late C. E. L. Brown, were shown. One such motor was a three-phase, 20 b.h.p. motor. It had a distributed stator winding and a squirrel-cage rotor, and a small air-gap. The squirrel-cage rotor was the invention of Mr. Dolivo-Dobrowolsky, who co-operated with Mr. Brown in the design of these motors. This motor of Brown's closely resembled in construction the motor of to-day. From the short account given, it will be realized that much progress was made purely as the result of experiment, and that much theoretical investigation was needed to explain the reactions taking place in the motor, and also to show how it could be designed to give the characteristics desired. To that end it was necessary to give a lucid theory of alternating currents. Thomas H. Blakesley gave a series of ten brilliant papers in the *Electrician* in 1885. He discussed, for the first time, alternating current phenomena by means of polar diagrams. Then followed the work of the late Professor Gisbert Kapp in his papers contributed to the Institute of Civil and Electrical Engineers in 1890.

In 1892, F. Bedell and A. C. Crehore published their book, *Alternating Currents*, in which polar diagrams were used and applied to the theory of the transformer and the locus of the primary e.m.f. of the current transformer was shown to be a circle. In 1894, Kapp gave a very lucid elementary account of the phenomena in induction motors. The polar diagram was developed and included the primary resistance and leakage. His diagram was given for each point of the load, and gave no general solution showing how the different characteristics varied with the load.

In 1895, Blondel gave his papers on "Some General Properties of Revolving Magnetic Fields." In these papers, published in *Eclairage Electrique*, he unfolded the theory of the composition of magnetic fluxes, including leakage fluxes. In 1895 also, the late Mr. B. A. Behrend proved that the locus of the primary current of the alternating-current transformer is a circle in the polar diagrams, provided the primary resultant magnetic field is constant. The circle locus of the induction motor was also shown by A. Heyland in 1894. There has been much controversy about priority in this discovery. Part of the credit must be given to Dr. Bedell, who stated: "In any circuit

or apparatus with constant reactance and variable power consumption, the current will have a circle locus if the supply voltage is constant." This was first shown by Bedell and Crehore in 1892.

Bedell also states: "That the induction motor nearly fulfils those conditions and that its current locus is practically the arc of a circle, was first shown by Heyland in 1894." It is also stated that Kapp and Behn-Eschenberg first pointed out the identity of the theory of the a.c. transformer and the induction motor in 1893 and 1894.

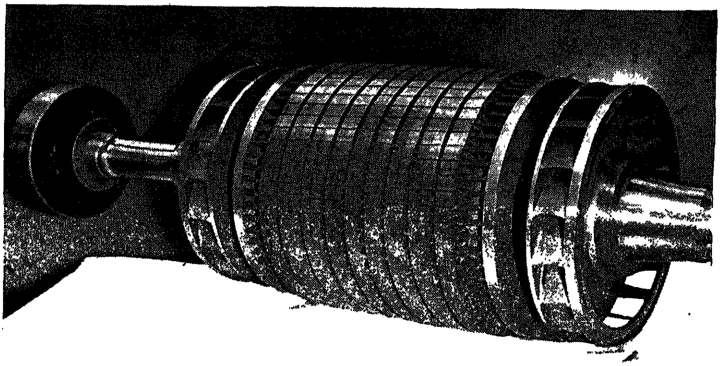
The circle diagram in use to-day is undoubtedly due to Behrend, and one is impressed with the beautiful simplicity of the diagram, and the ease with which the characteristics of the motor are determined from it, commends it to the designer. My first knowledge of this motor came from his stimulating book, the first edition of which was published in 1901. The second edition was published in 1921. Behrend was greatly interested in this motor and, when the first edition of this book appeared in 1925, he invited me to Boston and expressed great pleasure at my entry into this field, which he had made so much his own and to which he had contributed so much. I visited him in Boston. To me he was most gracious. I was impressed with his personality and especially with his modesty. He was a great friend of Heaviside, and I was surprised to see on the walls of his home photographs of Heaviside at various ages.

Even at that time the general outline was clearly seen, but there remained many important questions to be answered. The circle diagram clearly demonstrated the need for a small air-gap length, for high power factor, and a large ideal short-circuit current. All the main characteristics, such as torque, power, current, slip, efficiency are readily determined from the Behrend circle diagram. Behrend gave an empirical formula for the dispersion coefficient in the form $\sigma = C\frac{\delta}{\tau}$, where δ = gap length, τ = pole pitch, and C = a factor which varied with slot dimensions and other things.

This led some to assume that the best power factor was obtainable by using larger and larger ratios of diameter to core length for a given D^2L.

Closer analysis has shown that this is not true, and the exact relation of D to L for best power factor was given by me in my book in 1925.

Although the relation of dimensions to characteristics was known about 1900 or so, it remained to determine the effect of harmonics, due to the distribution of the winding, and also due to slots, on the performance of the motor. Analysis of the m.m.f. diagrams showed that several rotating fields were produced, the fundamental and various harmonics. These fields rotate at different speeds with respect to the rotor, and some rotate in the same direction, some in the opposite direction, and produce both driving and retarding torques. Such harmonics may, and do, produce noise and vibration,



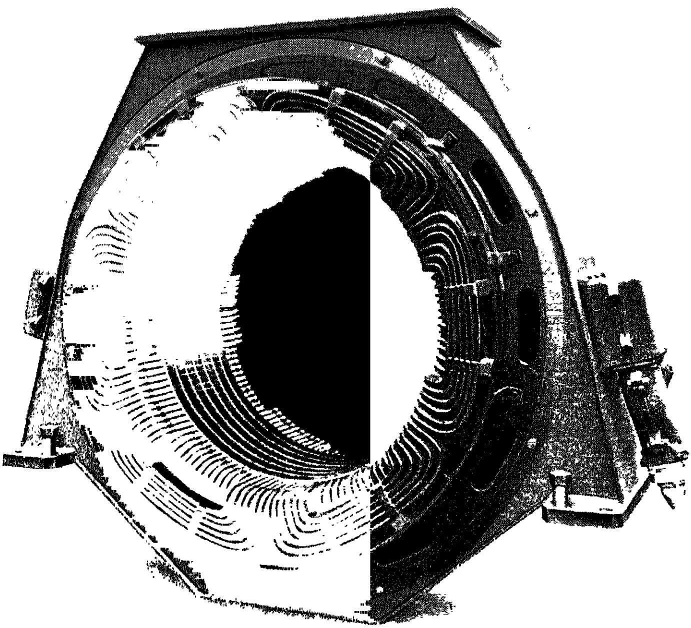


PLATE I

(*Upper*) SQUIRREL-CAGE ROTOR

(*Courtesy English Electric Co., Ltd.*)

(*Lower*) STATOR FOR COMMON TYPE OF INDUCTION MOTOR

(*Courtesy English Electric Co., Ltd.*)

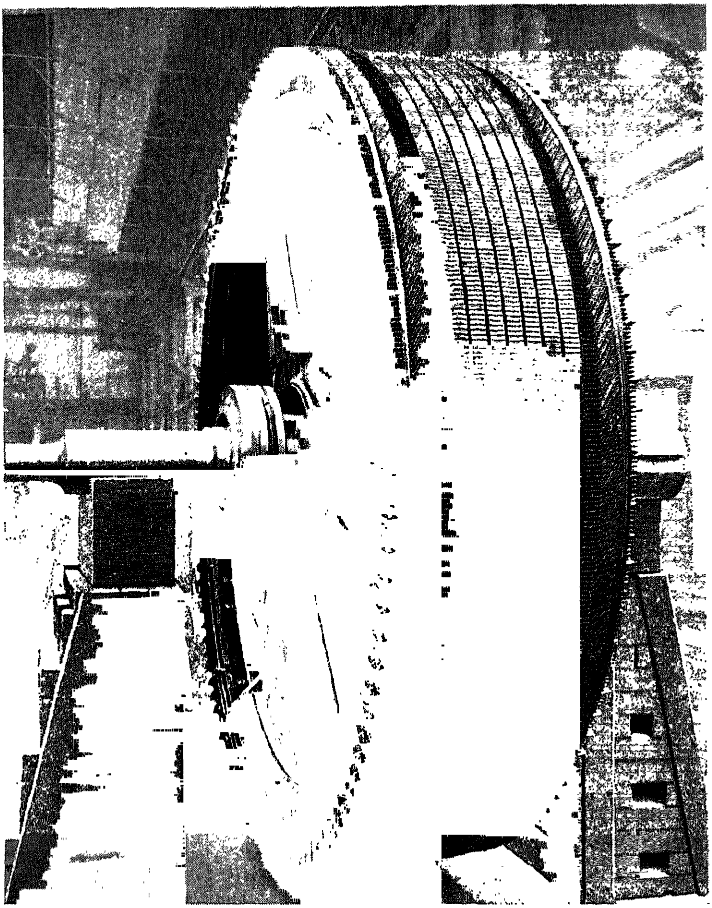


PLATE II. THREE-PHASE TYPE OF ROTOR

(Courtesy English Electric Co., Ltd.)

and may, by producing saddle backs in the torque curve, prevent the machine from accelerating to full speed. Especially is this true in pole-changing motors. This question of noise has been investigated by H. Fritze and Kron and Chapman. A large number of papers has been written on this very important subject.

The question of improved efficiency has resulted in improvements in the manufacture of steel laminations, by the introduction of silicon in various percentages. Brands known as Stalloy and Super-Stalloy have been introduced, and are used in those cases where it is necessary to keep down the iron losses, in totally enclosed machines, and especially in machines for 400 c/s, such as are used in connection with the automatic pilot for planes.

Then the question of eddy-current losses in conductors has been thoroughly investigated by A. B. Field and others.

The trend is towards greater and greater output from a given mass of materials, and this can only be effected by scientific design and proper proportioning of the machine. The introduction of Silicone varnishes for insulation has removed the conservative temperature rises formerly allowed, and resulted in smaller machines for a given output and speed.

Ventilation is another problem, which has received much attention and is one of the most important factors in increasing output from a given frame. Much more research is still needed on this important question.

The question of speed control in induction motors has received much attention. The induction motor is essentially a constant-speed machine, like the d.c. shunt motor, and this is, in some cases, rather a serious drawback. There are many industrial applications where speed control is necessary and the induction motor is the ideal motor for many such applications, being simple, rugged, and reliable, but various methods must be adopted to secure efficient speed control which spoil its simplicity. The methods adopted are: (1) pole-changing; (2) cascade connection; (3) cascading with a commutator motor; (4) change of frequency of supply. All these methods will be dealt with later, but the work of Mr. Louis J. Hunt deserves special mention for his genius and originality in producing the cascade motor bearing his name. I should like to pay him the greatest tribute, for I was associated with him in the early stages of his invention. His name should never be forgotten, for his work bears the stamp of genius.

The induction motor is now the most widely used of all machines. It is doubtful whether the large power systems, now in such extensive use, would have been developed if this motor had not been developed. What it means to the economy of the world is appreciated by few people outside the engineering world. In the fractional horse-power field, its development is phenomenal and it is applied in every form of industrial and domestic work. In this field it takes the form of

single-phase and three-phase types. We shall be dealing with the problems of single-phase capacitator motors in the text, and with their characteristics.

In analysing the performance of both three-phase and single-phase motors, we are frequently faced with unbalanced conditions, such as unbalanced voltages or unbalanced windings. These interesting problems will be solved by the use of the theory of symmetrical components, due largely to the late Dr. C. L. Fortescue. An explanatory account of the theory will be given in the text, and applications made to many interesting problems.

CHAPTER I

The Polyphase Induction Motor

THE induction motor is the most extensively used of all alternating-current motors. In its simplest form it admits of robust mechanical construction, and its ruggedness and ability to stand rough usage make it a most desirable type of industrial motor. It consists essentially of a stationary member, called the stator, and a rotating member called the rotor. The active part of the stator consists of a core of laminations of sheet steel of about 0·5 mm in thickness. These laminations are slotted on their inner periphery, and assembled in a steel yoke.

In these slots is placed a winding of the required number of phases, which may be of the concentric type, of the mush type, or of the barrel type with diamond-shaped coils. Plate I (lower) shows the stator for a common type of induction motor.

The rotor may be of the squirrel-cage type, consisting of bars of copper or aluminium placed in the rotor slots and connected at each end by a solid ring of copper, aluminium or brass. Plate I (upper) shows a normal type of squirrel-cage rotor.

If the starting requirements are such as to demand large starting torque with low starting current, then a rotor of the wound type with slip-rings will be used, and starting resistances will be used. This rotor is usually of the three-phase type with either mush or diamond coils. Plate II shows this type of rotor. While the stator may be of single-phase or of polyphase type, the rotor is always of the polyphase type, i.e. either two-phase or three-phase.

In the analysis of the magnetomotive force diagrams of the usual types of windings, which will be given in Chapter XI, it is shown that with symmetrical polyphase windings, a rotating field is produced when these windings are supplied with symmetrical polyphase currents. A symmetrical m phase winding is defined as one in which the starts of the phases are displaced around the periphery by $\frac{2\pi}{m}$ or $\frac{360}{m}$ electrical degrees. Each pole pitch corresponds to π electrical radians or 180 electrical degrees, and, of course, the double pole-pitch is 360 electrical degrees or 2π radians. Thus, the starts of the phases in a three-phase machine must be displaced around the periphery by two-thirds of the pole pitch or 120 electrical degrees,

i.e. $\frac{360}{3}$. The ends of the phases will be displaced by the same amount from each other. Now if we supply such a polyphase winding with alternating currents which vary sinusoidally with respect to time, and in which the time phase difference is $\frac{2\pi}{m}$, it will be shown that the magnetomotive forces of such polyphase windings can be represented by a series of rotating waves, each of constant amplitude, rotating at speeds which vary inversely as their order. The principal wave, the fundamental, rotates at synchronous speed, i.e. at $\frac{120f}{p}$ r.p.m. The fundamental is of constant amplitude and rotates at synchronous speed around the machine. The smaller waves are called "harmonics." In the non-chorded windings and also the chorded windings which can be replaced by non-chorded windings, no even harmonics are present. In these three-phase windings we have odd harmonics as follows: $1, -5, +7, -11, +13, -17, +19$, etc.

The minus sign indicates that these harmonics travel in the opposite direction to the fundamental wave. Thus, the 5th, 11th, 17th harmonics, etc., travel in the negative direction, while the 1st, 7th, 13th, 19th travel in the same direction as the fundamental. The orders of the various harmonics are given by the following equation—

$$\beta = \alpha_1 m_1 + 1$$

where α is a positive or negative integer and m_1 is the number of stator phases.

Thus, $\alpha_1 = 0$, $\alpha_1 = -2$; $\alpha_1 = +2$, $\alpha_1 = -4$, etc.; $\beta = +1$, $\beta = -5$; $\beta = +7$, $\beta = -11$.

The above values for β, viz. $1, -5, +7, -11, +13, -17, +19$, show the various harmonics for a non-chorded three-phase winding.

It will be noted that there is no third harmonic in the three-phase winding. The pole pitch for each harmonic is, of course, equal to the pole pitch for the fundamental divided by the order of the harmonic.

The fundamental fact is that, under the conditions stated, several revolving fields are produced in the machine, each of constant amplitude, and if the speed of the fundamental wave is R.P.M.$_s$, then the speeds of the harmonics are $\frac{\text{R.P.M.}_s}{\beta}$, where β = order of the harmonic.

Thus, the fifth harmonic rotates at one-fifth of the speed of the fundamental, the seventh at one-seventh of the speed of the fundamental, etc.

These rotating fields generate e.m.f.s in both stator and rotor windings. The e.m.f.s of the various harmonic fields generated in

the stator are all of fundamental frequency, given by the usual equation—

$$\frac{\text{poles}}{2} \times \text{r.p.s.} = \text{frequency}$$

It will be clear that, since the number of poles for the various harmonics vary directly as their order, and the speeds vary inversely as their order, the generated frequency of the e.m.f.s in the stator are all equal.

There are other harmonics present, in addition to those mentioned above, which are introduced by the rotor and stator slots. All these rotating fields are responsible for e.m.f.s generated in both stator and rotor windings and their resulting currents. By the interaction of fluxes and currents, driving and retarding torques will be produced, which may affect the characteristics of the motor. It will be our purpose to study the effects of these harmonics in a later section, and the steps which may be taken to reduce or eliminate their effects. At present it is our object to get a clear understanding of the reactions taking place in the machine and its manner of working.

We will suppose there is a symmetrical three-phase winding supplied with three-phase alternating current, and we will suppose that the fundamental wave of rotating flux only is present. The direction of rotation of the fundamental wave will depend on the sequence of the phases. It will be shown that the revolving field amplitude always lies above the winding group in which the current is a maximum, i.e. the sequence in which the current reaches its maximum in the various phases determines the direction of rotation. Thus, by reversing the connection of two of the terminals, the direction of the rotation may be reversed. Assume that a rotor with a three-phase symmetrical winding exists. The revolving field will generate e.m.f.s in stator and in rotor. At standstill these e.m.f.s are of the same frequency. If the rotor circuits are closed, currents will be set up in the rotor circuits of the same frequency as the rotor e.m.f.s. Since, at standstill, the frequency of the rotor currents is equal to the supply frequency, the leakage reactance of the rotor per phase will be relatively high compared to the resistance, and hence the angle of lag of the rotor current behind the e.m.f. and flux will be large, and this will be true for each phase.

Now we know that conductors carrying current in a magnetic field are subjected to a force or torque, and this force can be calculated for each conductor of the rotor, once we know the current in the conductor and the value of the field density in which it lies. Let us assume the field rotates in the clockwise direction as shown below in Fig. 1.1, and let us assume the flux is distributed sinusoidally.

The direction of the e.m.f.s and currents in the rotor conductors is shown by the dots and crosses. Now, if one places the *left* hand

along the conductors in such a manner that the flux passes from the palm to the back, and the current flows from the wrist to the finger tips, then the direction of the force on the conductors is towards the right as shown by the direction in which the thumb points. The rotor, therefore, moves in the direction of movement of the field. It should be clear that the torque at starting is reduced by the angle

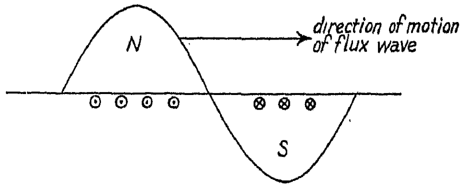


FIG. 1.1

of lag of the rotor current behind its e.m.f. Now, if this lag could be reduced, the starting torque would be increased. This is usually effected by adding resistance in series with the rotor.

Fig. 1.2 shows a sketch of the three-phase wound rotor motor with starting resistance.

Since $$\tan \phi_2 = \frac{\text{rotor reactance per phase}}{\text{rotor resistance per phase}}$$

it follows that the lag angle can be reduced by the addition of resistance in series with each phase of the rotor. The rotor therefore

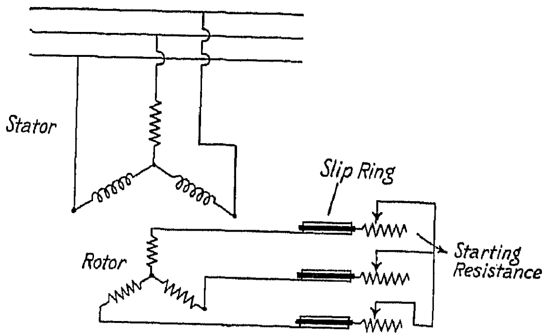


FIG. 1.2

accelerates, and as its speed increases, the rate of cutting of the flux by the rotor conductors decreases and is dependent on the relative velocity of the flux and rotor.

If ω_0 = speed of the field in radians per second

ω = speed of the rotor in radians per second.

Then the relative speed $= \omega_0 - \omega$; and this expressed as a fraction of ω_0, the synchronous speed, is called the slip. If s = the slip, $s = \dfrac{\omega_0 - \omega}{\omega_0}$

Since the rotor frequency = pairs of poles × slip revolutions per second, it is clear that the frequency of the rotor currents = $s \times f$, where f = supply frequency.

At standstill $s = 1$, since $\omega = 0$ at standstill.

At synchronous speed $\omega = \omega_0$ and $s = 0$. Thus the frequency of the rotor currents varies from f at standstill, to sf at slip s.

The speed will continue to increase until the driving torque equals the resisting torque. With no load on the machine, the speed rises to a value not far from synchronous speed. The slip at no-load is a fraction of 1 per cent.

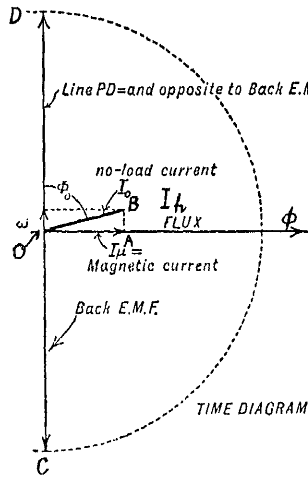


FIG. 1.3. VECTOR DIAGRAM AT NO-LOAD

Vector Diagram at No-load

The current in the rotor at no-load being relatively small, the current in the stator is that necessary to produce the flux. This flux is in time phase with the current in the stator winding, neglecting magnetic hysteresis. The e.m.f. generated in one phase lags 90° behind the flux linking the winding, and is in phase with the rotating flux cutting the conductors. Thus when the flux linking a coil is a maximum, the e.m.f. is zero.

The vector diagram at no-load is shown in Fig. 1.3.

OA = vector of flux

$OA = I_\mu$ = magnetizing current per phase

$AB = I_h$ = watt component of no-load current per phase

$OB = I_0$ = no-load current per phase

OC = back e.m.f. in the stator

OD = applied p.d., sensibly equal to OC, per phase

$\cos \widehat{DOB} = \Phi_0$ = power factor at no-load.

The no-load current per phase consists of a magnetizing component OA, and a watt component AB, per phase, to supply the no-load losses.

On-load Vector Diagram

Under full load the motor will have a slip such that sufficient e.m.f. will be generated, per phase, in the rotor to produce sufficient torque to equal the retarding torque due to the load and the friction and windage losses. Fig. 1.4 shows the on-load vector diagram.

It will be observed from the diagram, that the current flows in the rotor in such a direction as to tend to demagnetize the stator. In other words the amp-turns of the rotor winding tend to oppose the amp-turns of the stator. At no-load the applied p.d. per phase

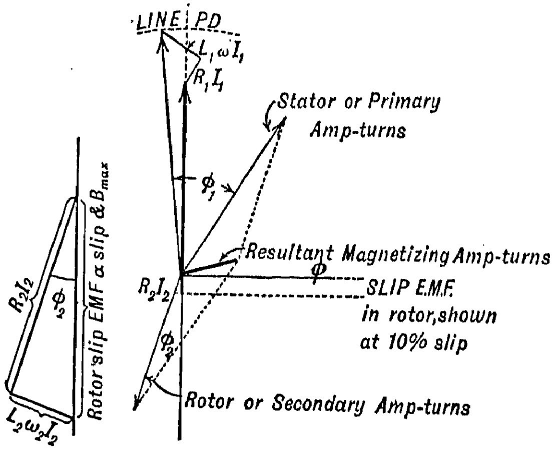


FIG. 1.4. VECTOR DIAGRAM ON-LOAD

is sensibly equal to the back e.m.f. generated in the stator by the revolving field.

Actually we have

$$\boldsymbol{V} = \boldsymbol{E} + \boldsymbol{Z}_s \boldsymbol{I}_{os} = \boldsymbol{E} + (r + jx_s)\boldsymbol{I}_{os} \quad . \quad . \quad (1.1)$$

where $\boldsymbol{V}$ = applied volts per phase
$-\boldsymbol{E}$ = back e.m.f. per phase
$\boldsymbol{Z}_s$ = leakage impedance per phase
$= r + jx_s$
r = stator resistance per phase
x_s = leakage reactance per phase of the stator

On-load also—

$$\boldsymbol{V} = \boldsymbol{E}_1 + (r + jx_s)\boldsymbol{I}$$

where $-\boldsymbol{E}_1$ = back e.m.f. on-load
$\boldsymbol{I}$ = current per phase in stator on-load

The addition in these equations is vectorial, vector quantities being indicated in bold-face type.

Virtually there is only a 3 per cent difference in the flux from no-load to full load, so we can assume the flux remains sensibly constant. For this to be so, it follows there must flow in the stator a component of current which offsets, by its magnetizing action, the demagnetizing effect of the rotor current.

The flux in the machine is produced by the resultant of the stator and rotor magnetomotive forces. Since the flux exists almost at the same value at no-load, it follows that the stator current on-load consists of two components, namely, the no-load current I_0, and the load component of the current.

The effective ampere-turns of the rotor = effective ampere-turns in the stator, due to the load component of the current.

The e.m.f.s in rotor and stator will have the same relation to the flux as at no-load. Now, however, the rotor carries current lagging behind the e.m.f., and this lag is produced by leakage fluxes. This lag will depend also on frequency of the rotor currents which is proportional to the slip. At standstill this lag is important, for the slip is high, and the angle large. At full-load speed the slip may be anything from 2 to, say, 5 per cent, and the lag is small. As in a transformer this lag is reflected into the stator circuit, so at standstill the power factor is low, increasing as the slip decreases from 1 to the full-load value. Plates III and IV (facing pages 20 and 21) show examples of brush-lifting and short-circuiting gear and slip-rings.

Circle Diagram of the Motor

Let us consider a three-phase motor, and let—

T_1 = turns per stator phase in series
T_2 = turns per rotor phase in series
$\hat{\phi}$ = maximum value of flux per pole
f = supply frequency
s = slip
$\bar{E}_1$ = back e.m.f. in stator per phase
$\bar{E}_2$ = e.m.f. generated in the rotor per phase at standstill
K_1 = breadth factor of stator winding for fundamental
K_2 = breadth factor of rotor winding for the fundamental
K_3 = coil span factor of the stator winding for the fundamental
K_4 = coil span factor of the rotor winding for the fundamental

Then
$$\bar{E}_1 = 4{\cdot}44 \times K_1 \times K_3 \times \hat{\phi} \times T_1 \times f \times 10^{-8} \quad . \quad (1.2)$$
$$\bar{E}_2 = 4{\cdot}44 \times K_2 \times K_4 \times \hat{\phi} \times T_2 \times f \times 10^{-8} \quad . \quad (1.3)$$

The e.m.f. per phase in the rotor, at slip s, $= s\bar{E}_2$—

$$\therefore \quad s\bar{E}_2 = 4{\cdot}44 \times K_2 \times K_4 \times \hat{\phi} \times T_2 \times sf \times 10^{-8} \quad . \quad (1.4)$$

If L_2 = coefficient of self inductance of the rotor winding per phase in henrys, due to *leakage* flux

and $\omega_0 = 2\pi \times f$

and $\bar{I}_2$ = rotor current per phase in amperes

then $$s\bar{E}_2 = \bar{I}_2\sqrt{R_2^2 + s^2L_2^2\omega_0^2} \quad .$$

and $$\bar{I}_2 = \frac{s\bar{E}_2}{\sqrt{R_2^2 + s^2L_2^2\omega_0^2}} = \frac{\bar{E}_2}{\sqrt{\frac{R_2^2}{s^2} + L_2^2\omega_0^2}} \quad .$$

where R_2 = rotor resistance per phase.

Equation (1.6) shows that the rotor current is equal to th generated per phase at standstill, divided by $\sqrt{\left(\frac{R_2}{s}\right)^2 + L_2^2 \omega}$

Now $L_2\omega_0$ is the reactance per phase of the rotor at full frequency, i.e. at standstill.

We see, therefore, that the rotor current at standstill, provi total rotor resistance $= \frac{R_2}{s}$ = the actual rotor resistance pe divided by the slip, is equal to the rotor current when runnin load with slip s, and it is obvious that the phase relation of t current $\bar{I}_2$ to the rotor e.m.f. is the same in each case, fo $= \frac{sL_2\omega_0}{R_2}$ in each case.

Therefore, as far as current and power-factor relati concerned, the action of the machine, when running normall load, is exactly the same as at rest, provided we make the ne tance in the rotor circuit per phase $= \frac{R_2}{s}$.

Now $$\frac{R_2}{s} = R_2 + R_2\frac{(1 - s)}{s} \quad .$$

Now if we multiply each side of equation (1.7) by $\bar{I}_2^2$, we

$$\bar{I}_2^2\frac{R_2}{s} = \bar{I}_2^2R_2 + \bar{I}_2^2R_2\frac{(1 - s)}{s}. \quad .$$

Substituting for $\bar{I}_2$ its value from equation (1.6)—

$$\frac{s\bar{E}_2 \times \bar{I}_2}{\sqrt{R_2^2 + s^2L_2^2\omega_0^2}} \times \frac{R_2}{s} = \bar{I}_2^2R_2 + \bar{I}_2^2R_2\frac{(1 - s)}{s}$$

i.e. $$\bar{E}_2 \times \bar{I}_2 \times \cos\phi_2 = \bar{I}_2^2R_2 + \bar{I}_2^2R_2\frac{(1 - s)}{s} \quad .$$

Now considering the two terms on the right of equatior we have $\bar{I}_2^2R_2$, which is the rotor copper loss per phase a second term, namely, $\bar{I}_2^2R_2\frac{(1 - s)}{s}$ represents the additional phase which is equivalent to the gross output of the machin running under load with a rotor current per phase $\bar{I}_2$ at slip

With a three-phase rotor—

Gross mechanical output $= 3\bar{I}_2^2R_2\frac{(1 - s)}{s}$ in watts.

With an m-phase rotor, where m is an integer greater than 1, the gross mechanical output $= m \times \bar{I}_2{}^2R_2 \dfrac{(1-s)}{s}$ in watts.

We see, therefore, by making the resistance in the rotor circuit per phase $= \dfrac{R_2}{s}$, the machine is brought to rest and the current, power, and power-factor relations in the rotor are the same as when running under load with a slip s.

Thus, at rest, the motor becomes a transformer; a rather leaky transformer, and all the theory of the constant potential transformer can be directly applied to it.

Now looking at equations (1.2) and (1.3), we see that—

$$\bar{E}_2 = \frac{\bar{E}_1 \times T_2 \times K_2 \times K_4}{T_1 \times K_1 \times K_3} \quad . \quad . \quad . \quad (1.12)$$

We have also shown that $\bar{I}$ consists of two components, the no-load current and a component which is called the load component. The effective component of the stator current which neutralizes the magnetizing action of the rotor current is—

$$\frac{I_2}{\alpha} = I_2 \times \frac{T_2 \times K_2 \times K_4}{T_1 \times K_1 \times K_3} = \frac{I_2}{\alpha} = \bar{I}_1{}' \quad . \quad . \quad (1.13)$$

where $\qquad \alpha = \text{ratio of transformation} = \dfrac{T_1 \times K_1 \times K_3}{T_2 \times K_2 \times K_4} \quad . \quad (1.14)$

Now—

$$\bar{I}_1{}' = \frac{\bar{I}_2}{\alpha} = \frac{\bar{E}_2}{\alpha\sqrt{\left(\dfrac{R_2}{s}\right)^2 + L_2{}^2\omega_0{}^2}} = \frac{\bar{E}_1}{\alpha^2\sqrt{\left(\dfrac{R_2}{s}\right)^2 + L_2{}^2\omega_0{}^2}} \quad (1.15)$$

$$= \frac{\bar{E}_1}{\sqrt{\left(\dfrac{R_2\alpha^2}{s}\right)^2 + (L_2\omega_0\alpha^2)^2}} \quad . \quad . \quad . \quad . \quad . \quad (1.16)$$

Thus, if we multiply R_2 and L_2, each by α^2, i.e. by $\left(\dfrac{T_1 \times K_1 \times K_3}{T_2 \times K_2 \times K_4}\right)^2$, we obtain the component of the stator current corresponding to the load, namely, $\bar{I}_1{}'$.

It is clear also that—

$$(\bar{I}_1{}')^2R_2{}' = \frac{\bar{I}_2{}^2}{\alpha^2} \times R_2 \times \alpha^2$$

$$= \bar{I}_2{}^2 \times R_2 \quad . \quad . \quad . \quad . \quad (1.17)$$

where $R_2{}' = R_2 \times \alpha^2$, the rotor resistance referred to the stator per phase

and $\quad L_2{}' = L_2 \times \alpha^2$, the rotor leakage self-inductance referred to the stator per phase.

We can now build up our equivalent circuit of the motor, in which all quantities are referred to the stator (Fig. 1.5).

Across AB we have the applied voltage per phase $\bar{V}_1$.

Clearly $$\bar{V}_1 = (R_1 + jL_1\omega_0)\bar{I}_1 + \bar{E}_1 \quad . \quad . \quad . \quad (1.18)$$

Note $-\bar{E}_1$ = back e.m.f.

$+\bar{E}_1$ = component of applied P.D. to overcome E_1.

Clearly also the voltage across $CD = +\bar{E}_1$. This is also the voltage, which is applied to rotor circuit per phase, when all quantities of rotor resistance and reactance are referred to the stator.

Across CD we have connected one phase of the rotor, in which all quantities are referred to the stator by multiplying them by the square of the ratio of transformation. We see that we have the rotor

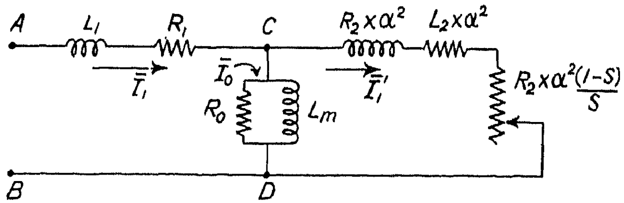


FIG. 1.5

resistance referred, namely, $R_2 \times \alpha^2$, and the rotor inductance referred, namely, $L_2 \times \alpha^2$, and the resistance representing the gross mechanical load also referred, namely, $R_2 \times \alpha^2 \dfrac{(1 - s)}{s}$. These are shown in series across the points C and D, between which we have the voltage $+\bar{E}_1$, i.e. the voltage which overcomes the back e.m.f. in the stator. It should be remarked that in all polyphase circuits, which are symmetrical, the quantities we require to estimate are the same for *each* phase, and hence it is sufficient to consider *one* phase only. Our diagrams refer to *one* phase only.

The back e.m.f. $-\bar{E}_1$ is produced by the flux ϕ per pole, and this requires a current for its production. It is the magnetizing current $\bar{I}_\mu$, shown in our vector diagram. It leads $-\bar{E}_1$ by 90°. Hysteresis and eddy-current losses are produced by the revolving field in the core of the stator and also in the rotor and also eddy-current losses in the conductors of the machine. A watt component of the current is needed to supply these losses. This component, namely $\bar{I}_h$, is in phase with $+\bar{E}_1$, i.e. with the component of applied volts across CD. Across CD is connected a purely inductive coil, whose reactance at supply frequency $= X_m$.

Then $$\bar{I}_\mu = \frac{+\bar{E}_1}{X_m}$$

In parallel with it is connected a non-inductive resistance R_0.

Its magnitude $= \dfrac{\bar{E}_1}{\bar{I}_h}$, i.e. it is of such a value as to give the watt component of current $\bar{I}_h$. The product of $\bar{E}_1$ and $\bar{I}_h$ gives the F and W losses, no-load copper loss + iron losses. The vector sum of $\bar{I}_\mu$ and $\bar{I}_h$ gives $\bar{I}_0$, the no-load current.

The current flowing through the resistance and leakage reactance of the stator is the vector sum of $\bar{I}_0$ and I_1' and equals $\bar{I}_1$. The resistance of the stator per phase $= R_1$ and its *leakage* coefficient of self inductance is L_1. It will simplify the results somewhat if we assume that the circuit connected across C and D, namely, L_m and R_0, is removed to the terminals A and B. This is equivalent to assuming the voltage across them is constant and the flux, therefore, constant. Actually the voltage across them varies slightly, due to the leakage impedance drop in R_1 and L_1. The circuit in Fig. 1.5 can, however, be reduced to a series circuit, with an effective resistance and effective reactance in series across the mains. Assuming, therefore, L_m and R_0 to be transferred across A and B, we have—

$$\bar{I}_1' = \frac{\bar{V}}{\sqrt{\left(R_1 + \dfrac{R_2'}{s}\right)^2 + (L_1 + L_2')^2\omega_0^2}} \qquad (1.19)$$

where $R_2' = R_2 \times \alpha^2$ and $L_2' = L_2 \times \alpha^2$.

The current $\bar{I}_1'$ = rotor current referred to the stator—

$$\bar{I}_1' = \frac{\bar{V}}{(L_1 + L_2')\omega_0} \times \frac{(L_1 + L_2')\omega_0}{\sqrt{\left(R_1 + \dfrac{R_2'}{s}\right)^2 + (L_1 + L_2')^2\omega_0^2}} \qquad (1.20)$$

but

$$\frac{(L_1 + L_2')\omega_0}{\sqrt{\left(R_1 + \dfrac{R_2'}{s}\right)^2 + (L_1 + L_2')^2\omega_0^2}} = \frac{\text{reactance}}{\text{impedance}} \qquad (1.21)$$

$$= \sin\phi$$

$$\therefore \qquad \bar{I}_1' = \frac{\bar{V}}{(L_1 + L_2')\omega_0}\sin\phi \qquad (1.22)$$

$$= I_{sci}\sin\phi \qquad (1.23)$$

It is clear that $\bar{I}_1'$ is represented by the chord in the semicircle OB, for $OB = OA \sin\phi$ (Fig. 1.6).

$$OA = \frac{\bar{V}}{(L_1 + L_2')\omega_0}$$

Equation (1.23) obviously represents the polar equation of a circle.

It will be noted that the diameter of the circle

$$= \frac{\text{applied volts per phase}}{\text{effective reactance per phase}}$$

In any purely reactive coil, the current will lag by 90° behind the volts across the coil. The current represented by OA in Fig. 1.6 is drawn lagging 90° behind V, for its value $= \frac{\bar{V}}{(L_1 + L_2')\omega_0}$, and is determined by the effective inductance of the machine referred to the stator, i.e. $(L_1 + L_2')$.

The current OA is called the ideal short-circuit current, for it is the current per phase in the stator which would flow at standstill if the machine possessed reactance only.

As the load changes on the motor, the *load* component of the stator current $I_1' = \frac{I_2}{\alpha}$ moves over the circle and is represented by OB in Fig. 1.6. Since the motor possesses resistance also, the actual

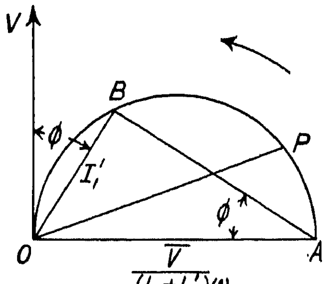


FIG. 1.6

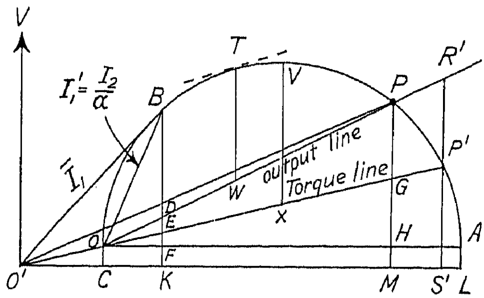


FIG. 1.7

load component of the stator current at standstill will be represented by a vector OP, where—

$$OP = \frac{\text{applied volts per phase}}{\text{effective impedance of the machine at standstill}}$$

i.e. $OP =$ actual load component of the stator current at standstill

$$= \frac{\bar{V}}{\sqrt{(R_1 + R_2')^2 + (L_1 + L_2')^2\omega_0^2}} \quad . \quad . \quad . \quad . \quad (1.24)$$

And the power factor $(s = 1)$ at *standstill*

$$= \cos \phi_s$$

$$= \frac{R_1 + R_2'}{\sqrt{(R_1 + R_2')^2 + (L_1 + L_2')^2\omega_0^2}} \quad . \quad . \quad (1.25)$$

The total primary current on load is obtained by adding vectorially the no-load current I_0 to I_1'.

Now, draw OC downwards (Fig. 1.7), parallel to $O'V$ and make $OC =$ watt component of the stator current per phase necessary to

supply the no-load losses. The diagram refers to one phase only and all quantities on the diagram are phase quantities.

$$OC = \frac{\text{total iron losses} + \text{friction and windage loss} + \text{no-load copper losses}}{\text{number of stator phases} \times \text{volts per phase on stator}}$$

The no-load losses are assumed to remain constant. This, of course, is not true. The flux per pole decreases on-load, due to the impedance drop in the stator, and actually the back e.m.f. E_1 decreases for $\mathbf{V} = +\mathbf{E}_1 + Z_1\mathbf{I}_1$ (vector addition). Therefore, the iron losses in the stator decrease slightly, but as the slip increases with the load, the iron losses in the rotor increase, for they depend on the slip frequency. The friction and windage loss fall as the speed falls, i.e. when the slip increases. No important error is made by assuming the no-load losses to remain constant, i.e. OC is assumed constant. Now set off CO' parallel to OA and make it equal to the magnetizing current per phase. Then $O'O$ is the no-load current. We thus transfer our origin of vectors from O to O'.

The point O on the circle represents the no-load point and the point P the short-circuit point. Draw from any point on the circle lines perpendicular to $O'L$. The stator current per phase is represented by lines drawn from the origin O' to the point in question.

At B, which represents some point on the circle corresponding to a certain load, $O'B$ is the stator current per phase. $O'B$ can be resolved into a component $O'K$, which is a wattless current, and BK, which is a watt current. It will be noted that, as the load increases, and the point B moves over the circle to the right, the wattless component of the stator current increases from $O'C$, at no-load, to $O'M$ at standstill. This is due to the increase of stator and rotor leakage flux, which increases as the currents in stator and rotor increase. The watt component of $O'B$, i.e. BK represents the power input to the stator. The total power input at the point $B = m_1 \times V \times BK$, where m_1 = number of stator phases. The stator power factor at point $B = \cos BO'V$.

It is, perhaps, desirable at this point to notice that high power factor, at any load, depends on making $O'K$ as small as possible; that is, the magnetizing current per phase, $O'C$, must be as small as possible, and also the leakage fluxes of stator and rotor must be kept small. This means a small air-gap length, for the magnetizing current depends on the length of the air-gap, and there must be no saturation in the iron part of the circuit. The leakage fluxes will be discussed in a later chapter. At no-load the leakage fluxes are small, for the current is small, and so virtually all the flux per pole crosses the air-gap and enters the rotor. At standstill the leakage fluxes are large, and most of the flux per pole exists as leakage flux, i.e. flux which follows local paths and does not follow the useful path.

It will also be clear that maximum power factor is obtained when

$O'B$ is tangent to the semicircle, the maximum power factor being determined by the cosine of the angle the tangent to the circle from O' makes with $O'V$.

It is also clear that the maximum input to the motor is determined by the radius of the semicircle. Thus, it is important to make the effective leakage reactance of the machine low for high short-circuit current. Now at P, $O'P$ is the primary current, and its watt component is PM, which is the power component at standstill. The power input, at standstill, which is equal to $\bar{V} \times PM$ per phase is absorbed in stator and rotor copper losses. PH represents the copper losses due to the component of current OP. When the load component of the stator current is OB, it will be shown that the copper losses in stator and rotor are represented by DF, for $\frac{DF}{PH} = \frac{OB^2}{OP^2}$

For
$$\frac{DF}{PH} = \frac{OF}{OH} = \frac{OB \cos BOF}{OP \cos POH} \quad . \quad . \quad . \quad (1.26)$$

$$\cos BOF = \frac{OB}{OA} \text{ and } \cos POH = \frac{OP}{OA} \quad . \quad . \quad (1.27)$$

$$\therefore \quad \frac{DF}{PH} = \frac{OB^2}{OP^2} \quad . \quad . \quad . \quad . \quad (1.28)$$

Now let GH represent the watt component of the *stator* current per phase corresponding to the stator copper losses with primary current OP at short-circuit.

The PG must represent the watt component corresponding to the *rotor* copper loss per phase at standstill.

Join OG, then EF represents the watt component of current corresponding to the stator copper loss per phase with the load component OB, and DE represents the rotor copper loss per phase under the same conditions.

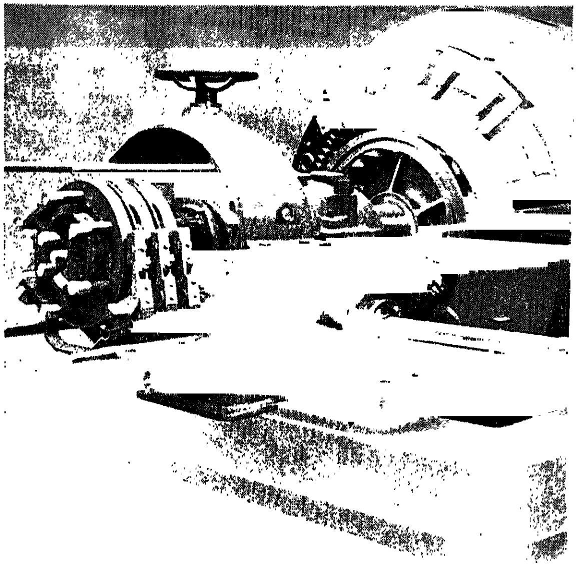
Since the watt input current per phase with current $O'B = BK$, FK corresponds to the no-load losses per phase, EF corresponds to the stator copper loss per phase with the load component OB, DE corresponds to the rotor copper loss per phase, and BD must represent the output.

Summing up, we have, with current $O'B$ in the stator—

$m_1 \bar{V} \times BK$ = input to motor in watts (total)
$m_1 \bar{V} \times BD$ = output in watts (total)
$m_1 \bar{V} \times DE$ = rotor copper losses (total)
$m_1 \bar{V} \times EF$ = stator copper losses due to load component of current OB

The power factor, when the stator current is $O'B = \cos BO'V$.

For any other value of the load, all we need to do is to draw a line from the point of the circle, corresponding to the load, perpendicular to OA, then the output is represented by the vertical intercept



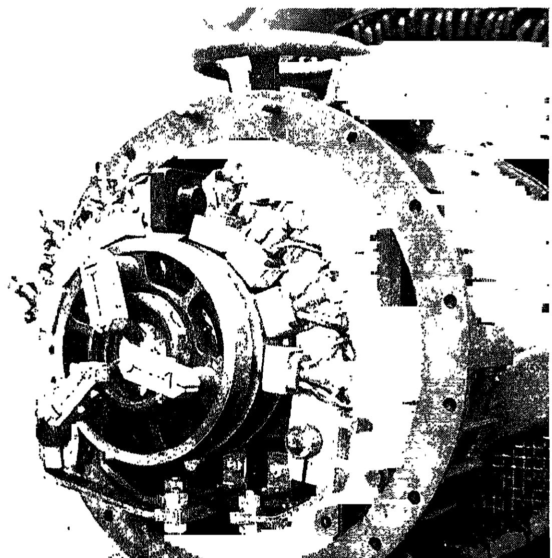


PLATE III

(*Upper*) BRUSH-LIFTING AND SHORT-CIRCUITING GEAR

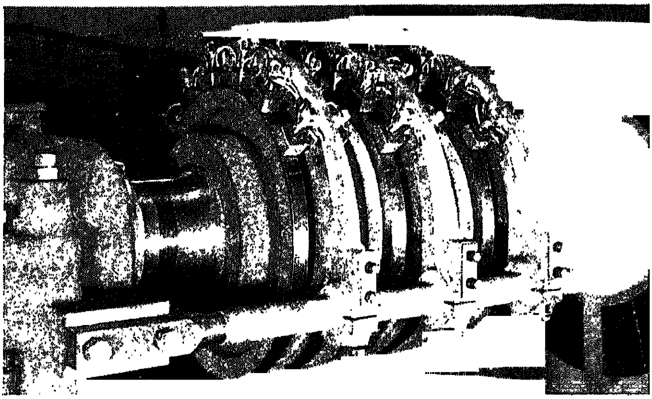
This is a 2430 h.p., 1500 r.p.m. motor. The end plate and enclosing cover have been removed

(*Courtesy English Electric Co., Ltd.*)

(*Lower*) CONTINUOUSLY-RATED EXTERNAL SLIP-RINGS WITH SHORT-CIRCUITING DEVICE ONLY

The enclosing cover has been removed

(*Courtesy English Electric Co., Ltd.*)



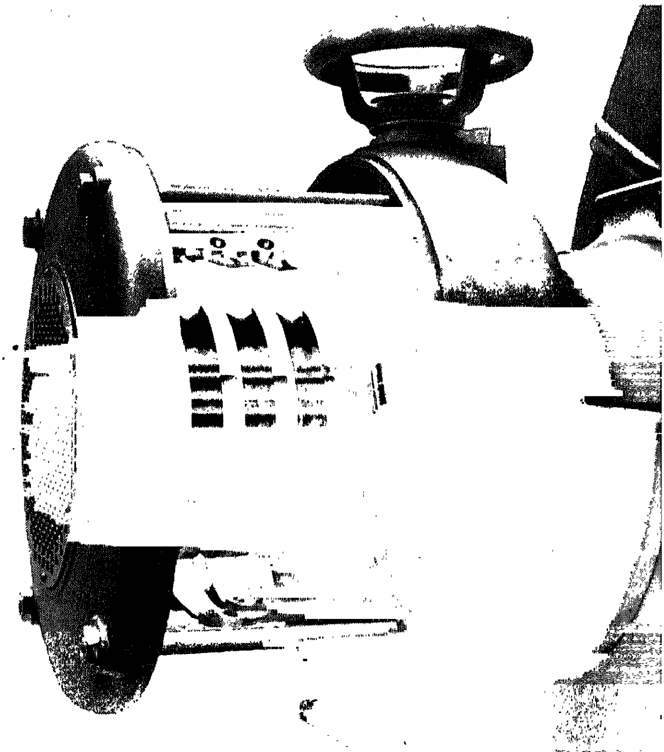


PLATE IV

(Upper) CONTINUOUSLY-RATED INTERNAL SLIP-RINGS FOR LARGE-OUTPUT I

(Courtesy English Electric Co., Ltd.)

(Lower) BRUSH-LIFTING AND SHORT-CIRCUITING GEAR ON EXTERNAL SLIP-RINGS OF MEDIUM-OUTPUT INDUCTION MOTOR

(Courtesy English Electric Co., Ltd.)

(T.591)

between the point on the circle and the line OP; the intercept between the lines OP and OG, made by this perpendicular, represents the rotor copper loss; and, similarly, the intercept, on the same line, made by the lines OG and OH represents the stator copper loss.

Since efficiency $= \dfrac{\text{output}}{\text{input}}$, it is equal to $\dfrac{BD}{BK}$

The slip is equal to the—

$$\frac{\text{rotor copper loss}}{\text{rotor input}} = \frac{DE}{BE} \quad . \quad . \quad . \quad (1.29)$$

for rotor input = output + rotor copper loss.

BE, therefore, represents the rotor input.

The line OP is the *output* line and OG represents the *torque* line, for it will be shown that the torque × synchronous speed, expressed in watts, equals the rotor input.

It should be stated that the locus of the primary current is a circle only provided the inductances do not vary with the current. Actually with large values of the currents, i.e. when the currents approach standstill values, the leakage flux paths may become saturated, and if this occurs L_1 and L_2' decrease, with the result that the current increases beyond the value obtained by assuming no saturation. The result is that the relation between applied volts per phase and short-circuit current is not linear, but bends up as shown in Fig. 1.8.

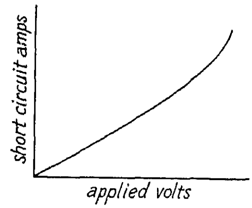


Fig. 1.8

This will obviously distort the circle diagram and give larger values for the starting torque than are calculated by assuming no saturation. Increase of eddy-current losses in conductors and iron due to large currents will increase the effective resistance, and this will tend to make the current curve decrease or bend down.

Torque and Slip Relations

We have shown that the input to the rotor per phase $= I_2^2 \dfrac{R_2}{s}$

The copper loss per phase $= I_2^2R_2$.

The difference is the output, namely, per phase

$$= I_2^2R_2 \frac{(1 - s)}{s} \quad . \quad . \quad . \quad . \quad (1.30)$$

If ω = angular velocity in radians per second

$= \omega_0(1 - s)$

and T = torque.

Then the output $= T\omega_0(1 - s) = m_2 \bar{I}_2^2 R_2 \left(\frac{1-s}{s}\right)$

$$\therefore \qquad T\omega_0 = \frac{\bar{I}_2^2 R_2}{s} \times m_2 \qquad . \qquad . \qquad . \quad (1.31)$$

m_2 = number of rotor phases.

Therefore, the torque, multiplied by the synchronous speed = the input to the rotor

$$= m_2 \bar{I}_2^2 \frac{R_2}{s} \text{ watts.} \qquad . \qquad . \qquad . \quad (1.32)$$

The torque, multiplied by the synchronous speed, expressed in watts, is usually referred to as "the torque in synchronous watts."

At standstill—

$$s = 1 \text{ and } T\omega_0 = m_2 \bar{I}_2^2 R_2 \quad . \qquad . \qquad . \quad (1.33)$$

Therefore, at standstill, the torque in synchronous watts = loss in the rotor due to copper losses.

Also, from equation (1.31), we see that the slip—

$$s = \frac{\text{rotor copper loss}}{\text{rotor input}} \qquad . \qquad . \qquad . \quad (1.34)$$

From equation (1.31) $T\omega_0 = \frac{\bar{I}_2^2 R_2}{s} \times m_2$

$$= m_2 \times \frac{s\bar{E}_2^2 \times R_2}{R_2^2 + s^2 L_2^2 \omega_0^2} \qquad . \quad (1.35)$$

where $\omega_0 = 2\pi f$.

For small values of the slip s—

$$T\omega_0 \doteqdot \frac{m_2 s \bar{E}_2^2}{R_2} \quad . \qquad . \qquad . \qquad . \quad (1.36)$$

i.e. for small values of the slip, the torque in synchronous watts is proportional to the slip, and for light loads and small s, the relation between slip and torque is linear.

For large loads and large slips—

$$T\omega_0 \doteqdot \frac{m_2 \bar{E}_2^2 R_2}{s L_2^2 \omega_0^2} \quad . \qquad . \qquad . \qquad . \quad (1.37)$$

i.e. the relation between torque and slip is a rectangular hyperbola.

It will be noted that the torque is proportional to the square of $\bar{E}_2$, i.e. proportional approximately to the square of the phase voltage applied, for $\bar{E}_2$ is proportional to $\bar{V}$.

Thus, it will be seen how important it is to see that the applied voltage is not low, when carrying out starting torque tests in which guarantees have been given.

It will be appreciated that the starting current with squirrel-cage rotors may vary from four to six times full-load current, and the power factor is low and there may be a large drop of voltage,

especially when the motor is supplied from an alternator of small or moderate capacity.

Maximum Torque and Slip at which it Occurs

The torque in synchronous watts $= \dfrac{m_2 s \bar{E}_2{}^2 R_2}{R_2{}^2 + s^2 X_2{}^2} = \tau$ . . (1.38)

where $X_2 = L_2\omega_0$.

$$\frac{d\tau}{ds} = \frac{m_2 \bar{E}_2{}^2 R_2\{R_2{}^2 + s^2X_2{}^2\} - 2sX_2{}^2(m_2 s\bar{E}_2{}^2R_2)}{(R_2{}^2 + s^2x_2{}^2)^2} \quad . \ (1.39)$$

For a maximum $\dfrac{d\tau}{ds} = 0$.

$$\therefore \quad R_2{}^2 + s^2X_2{}^2 = 2s^2X_2{}^2 \quad . \quad . \quad . \ (1.40)$$

$$\therefore \quad R_2{}^2 = s^2X_2{}^2 \quad . \quad . \quad . \quad . \ (1.41)$$

$$\therefore \quad s^2 = \frac{R_2{}^2}{X_2{}^2} \quad . \quad . \quad . \quad . \ (1.42)$$

$$\text{i.e.} \quad s_m = \pm\frac{R_2}{X_2} = \pm\frac{R_2}{L_2\omega_0} \quad . \quad . \quad . \ (1.43)$$

That maximum torque occurs when the slip—

$$s_m = \pm\frac{R_2}{L_2\omega_0} \quad . \quad . \quad . \quad . \ (1.44)$$

When $s_m = 1$, i.e. at standstill, maximum torque occurs when

rotor resistance per phase = rotor reactance per phase at standstill

Thus, it is possible, by using a wound rotor machine and external resistance, to satisfy this relation for maximum torque at starting.

The maximum torque is obtained by substituting for s the value $\dfrac{R_2}{X_2}$

Substitution for s in equation (1.38) gives—

$$\text{max. torque} = \frac{m\bar{E}_2{}^2}{2X_2} \quad . \quad . \quad . \ (1.45)$$

The maximum torque is independent of the *rotor* resistance, and is inversely proportional to the rotor reactance at standstill.

It will be seen, from Fig. 1.9, that the torque is plotted as function of the slip for various ratios of R_2 to X_2.

To obtain the necessary starting torque it is usual to use slip-ring type motors, when the large currents taken by squirrel-cage machines are prohibitive, and starting resistances are used to reduce the starting current. These resistances are cut-out as the speed rises.

Speed variation is also obtained by adding resistance to the rotor circuit as in rolling-mill motors, but it is obtained at greatly reduced efficiency, for the efficiency is always less than the speed as a percentage of synchronous speed.

The torque in synchronous watts = rotor input. With full-load current in the rotor at starting, we have full-load rotor loss. The slip $= \dfrac{\text{rotor copper loss}}{\text{rotor input}}$. The starting torque with full-load current, expressed as a percentage of full-load torque, is numerically equal to the slip at full load. The starting torque, expressed as a percentage of full-load torque, with a given current $\bar{I}$ in the rotor

$$= \left(\frac{\bar{I}}{\bar{I}_f}\right)^2 \times \text{per cent slip at full load} \quad . \quad . \quad (1.46)$$

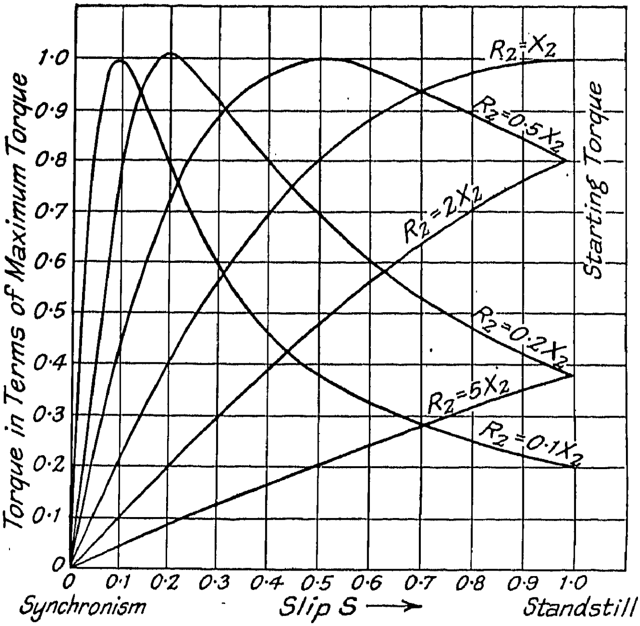


FIG. 1.9

where $\bar{I}_f$ = full-load rotor current, and the slip at full load

$$= \frac{\text{rotor copper loss at full load}}{\text{full-load torque in synchronous watts}}$$

The starting torque with rotor current $\bar{I}$ in synchronous watts = rotor copper loss with current $\bar{I}$.

Therefore, $\dfrac{\text{starting torque with rotor current } I}{\text{full-load torque with rotor current } I_f}$

$$= \frac{\text{rotor copper loss with current } I}{\text{rotor copper loss with current } I_f} \times \text{slip at full load}$$

$$= \left(\frac{I}{\bar{I}_f}\right)^2 \times \text{slip at full load} \quad . \quad . \quad . \quad . \quad . \quad (1.47)$$

The starting torque can be increased by increasing the current or by increasing the slip. A limit is set to the slip by rotor heating and the reduction in efficiency produced.

A large starting current of low power factor has a bad effect on the voltage regulation of the power system, and so for high starting torque a compromise must be made.

It is due to this effect on voltage regulation of the supply system that squirrel-cage motors are limited in size in Britain, and for outputs of over 5 b.h.p. it is usual to use rotors of the wound type, i.e. slip-ring type motors. In Canada and America this is not so and one finds the squirrel-cage type used for outputs of 200 h.p. and more. With the slip-ring type, full-load torque is obtained with approximately full-load current, and the cost of the energy used in starting is much less than with the squirrel-cage type. This may be an important matter where frequent starting of large numbers of motors is made.

The torque of a squirrel-cage machine is much more uniform than that of a wound rotor machine. The squirrel-cage conductors act powerfully to reduce flux variation and so tend to preserve a state of uniform torque.

With a wound rotor machine, the torque varies with varying position of the rotor with respect to the stator, due to varying impedance and varying value of zigzag leakage. It is important to keep the air-gap reluctance as constant as possible by choosing suitable slot ratios, and to keep the flux distribution as constant as possible both in value and in spacial distribution.

Construction for Slip

The slip $= \dfrac{\text{rotor copper loss}}{\text{rotor input}} = \dfrac{DE}{BE}$ in Fig. 1.7 for point B on the circle.

It is not easy to measure this ratio accurately from the diagram, so another construction is given in Fig. 1.10.

Join LP in Fig. 1.10 and produce LP to Q, and from Q draw QE at right angles to NG, the torque line. P is the standstill point.

Note: Q is any point on LP produced.

The triangles $B'NX$ and LCE are similar. So also are the triangles LQE and MNX.

$$\therefore \qquad \frac{B'X}{NX} = \frac{LE}{CE} \qquad (1.48)$$

and

$$\frac{LE}{EQ} = \frac{MX}{NX} \qquad (1.49)$$

$$\therefore \qquad CE = \frac{MX}{B'X} \times EQ \qquad (1.50)$$

The slip $= \dfrac{MX}{B'X}$; but this is proportional to CE, for QE is constant.

Therefore, CE is proportional to the slip for the primary current OB' in Fig. 1.10. Let QE represent 100 per cent slip, then the ratio $\frac{CE}{QE}$ gives directly to the same scale the percentage slip for the stator current OB'. C is the point of intersection of $B'L$ and QE. It is thus clear that, to obtain the slip directly, we must join the points on the circle representing the different primary currents to the point L. Then the intersection of these lines with QE gives the percentage

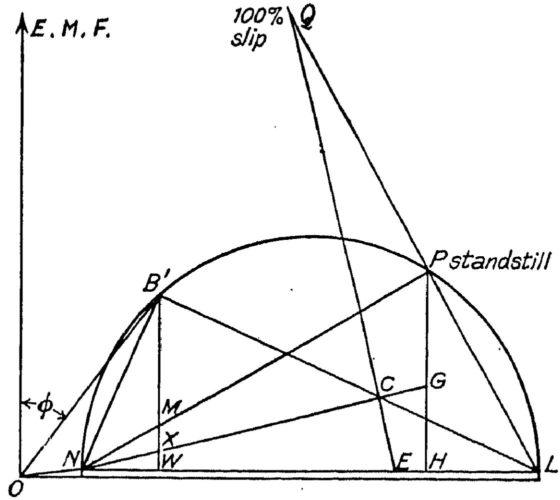


FIG. 1.10

slip directly, the percentage slip being given by the length of the line from the point of intersection to E.

It is interesting to note also that, at the point P', in Fig. 1.7, which is the point of intersection of the torque line OG with the circle, corresponds to infinite slip. $P'R'$ represents the mechanical power, as for any other point on the circle, but it is now a power *input*. This is used to supply the rotor copper loss. The primary power input, corresponding to $P'S'$, is used in primary copper losses, since the iron loss is small at this point.

Output and Slip for Maximum Output

$$\text{The torque, in synchronous watts} = \frac{m_2 s \bar{E}_2^2 R_2}{R_2^2 + s^2 X_2^2}$$

$$\text{The output} = \frac{m_2 s \bar{E}_2^2 R_2}{\omega_0 (R_2^2 + s^2 X_2^2)} \times \omega_0 (1 - s)$$

$$= \frac{m_2 \bar{E}_2^2 R_2 \times s(1 - s)}{R_2^2 + s^2 X_2^2} \text{ watts} \quad . \quad . \quad (1.51)$$

$$= \frac{C \times s(1 - s)}{R_2^2 + s^2 X_2^2}$$

where $C = m_2 \bar{E}_2^2 R_2$.

$$\frac{d\,(\text{output})}{ds} = \frac{C(1 - 2s)(R_2^2 + s^2X_2^2) - 2sX_2^2Cs(1 - s)}{R_2^2 + s^2X_2^2} \qquad (1.52)$$

for a maximum—

$$(1 - 2s)(R_2^2 + s^2X_2^2) = 2s^2X_2^2(1 - s)$$

i.e. $$R_2^2 + s^2X_2^2 - 2sR_2^2 - 2s^3X_2^2 = 2s^2X_2^2 - 2s^3X_2^2$$

i.e. $$R_2^2 - 2sR_2^2 = s^2X_2^2$$

$$\therefore \quad s = \frac{-2R_2^2 \pm \sqrt{4R_2^4 + 4R_2^2X_2^2}}{2X_2^2} \qquad (1.53)$$

$$= \frac{-R_2^2}{X_2^2} \pm \frac{R_2}{X_2}\sqrt{\frac{R_2^2}{X_2^2} + 1}$$

$$= -a^2 \pm a\sqrt{a^2 + 1} \qquad (1.54)$$

where $a = \dfrac{R_2}{X_2}$

The plus sign before the radical refers to motor action, the minus sign to generator action.

Equation (1.54) gives the slip at which maximum output occurs. To find the maximum output of the motor, we must substitute the value for s, obtained in equation (1.54), in equation (1.51).

We obtain for the maximum output—

$$\frac{m_2\bar{E}_2^2}{2X_2}\left(\sqrt{a^2 + 1} - a\right) \qquad (1.55)$$

$$= \frac{m_2\bar{E}_2^2}{2X_2^2}(Z_2 - R_2) \text{ watts} \qquad (1.56)$$

where Z_2 = impedance of the rotor per phase at standstill.

$$= \sqrt{R_2^2 + X_2^2} \qquad (1.57)$$

The following table shows the effect of rotor resistance on maximum output and also the slip at which maximum output and maximum torque occurs—

$a = \dfrac{R_2}{X_2}$	Maximum Output		Maximum Torque	
	Relative Magnitude of Maximum Output $\sqrt{a^2 + 1} - a$	Slip at Maximum Output	Relative Magnitude at Maximum Torque	Slip at τ_{max}
5	0·099	0·495	1	5
2	0·236	0·472	1	2
1	0·414	0·414	1	1
0·5	0·618	0·309	1	0·5
0·2	0·820	0·164	1	0·2
0·1	0·905	0·090	1	0·1

Fig. 1.11 shows the mechanical output at different speeds as a function of rotor resistance R_2.

In deducing the equations for maximum torque and maximum output, $\bar{E}_2$ was assumed constant. The equations can be calculated in terms of $\bar{V}$ the supply volts.

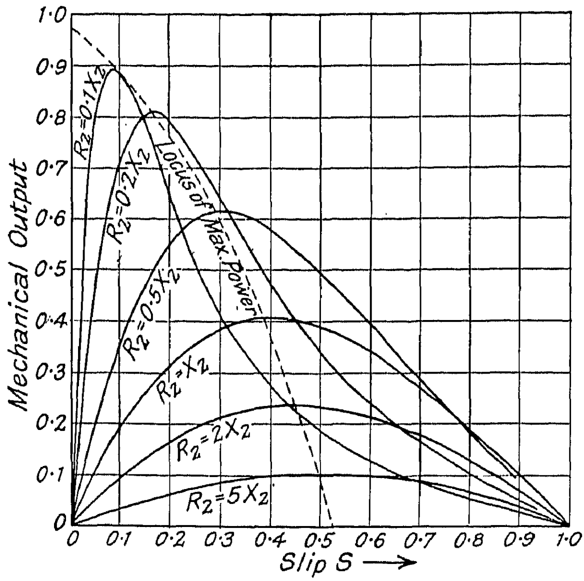


FIG. 1.11

Approximately

$$\boldsymbol{V} = \boldsymbol{I}_1' \left\{(r_1 + jx_1) + \left(\frac{r_2'}{s} + jx_2'\right)\right\} \quad . \quad . \quad (1.58)$$

$\boldsymbol{V}$ = supply volts per phase

r_1 and x_1 = stator resistance and reactance respectively per phase

r_2' = rotor resistance referred to the stator per phase

x_2' = rotor reactance referred to the stator per phase

$\boldsymbol{I}_1'$ = load component of stator current per phase

$$\boldsymbol{V} = \boldsymbol{I}_1' \left\{\left(r_1 + \frac{r_2'}{s}\right) + j(x_1 + x_2')\right\}$$

$$s\bar{V} = \bar{I}_1\sqrt{(r_1 s + r_2')^2 + s^2(x_1 + x_2')^2} \quad . \quad . \quad . \quad (1.59)$$

Also $\quad s\bar{E}_2 = \bar{I}_2\sqrt{r_2^{\,2} + s^2 x_2^{\,2}}$

$$= \frac{\bar{I}_1' \times \alpha\sqrt{(r_2')^2 + s^2(x_2')^2}}{\alpha^2}$$

and $\quad \alpha s\bar{E}_2 = \bar{I}_1'\sqrt{(r_2')^2 + s^2(x_2')^2} \quad . \quad . \quad . \quad (1.60)$

where α = ratio of transformation.

$$\frac{I_2}{\alpha} = I_1', \; r_2' = r_2 \times \alpha^2, \text{ and } x_2' = x_2 \times \alpha^2$$

$$\therefore \quad \bar{E}_2{}^2 = \frac{\bar{V}^2}{\alpha^2} \times \frac{((r_2')^2 + s^2(x_2')^2)}{(r_1 s + r_2')^2 + s^2(x_1 + x_2')^2} \quad . \quad . \quad (1.61)$$

The torque in synchronous watts—

$$= \frac{m_2 \times s\bar{E}_2{}^2 \times r_2}{r_2{}^2 + s^2 x_2{}^2}$$

$$= \frac{m_2 \times s\bar{E}_2{}^2 \times \dfrac{r_2'}{\alpha^2}}{\dfrac{((r_2')^2 + s^2(x_2')^2)}{\alpha^4}}$$

(m_2 assumed $= m_1$)

$$= \frac{m_2 \times \alpha^2 s\bar{E}_2{}^2 \times r_2'}{(r_2')^2 + s^2(x_2')^2} \quad . \quad . \quad . \quad (1.62)$$

Substituting from equation (1.61) for $\bar{E}_2$ in terms of $\bar{V}$, we have torque in synchronous watts

$$= \frac{m_2 \bar{V}^2 \times s r_2'}{(r_1 s + r_2')^2 + s^2(x_1 + x_2')^2} \quad . \quad . \quad (1.63)$$

and since the synchronous speed is 2π × synchronous revolutions per second—

$$\omega_0 = \frac{4\pi f}{p}$$

where p = number of poles.

The torque—

$$= \frac{m_2 \bar{V}^2 \times p \times s r_2'}{4\pi f\,(r_1 s + r_2')^2 + s^2(x_1 + x_2')^2} \quad . \quad . \quad (1.64)$$

If we differentiate equation (1.64) with regard to the slip and equate to zero, we get the slip at maximum torque

$$= \frac{r_2'}{\sqrt{r_1{}^2 + (x_1 + x_2')^2}} \quad . \quad . \quad . \quad (1.65)$$

and the maximum torque

$$= \frac{p\bar{V}^2}{4\pi f} \times \frac{m_2}{2\{r_1 + \sqrt{r_1{}^2 + (x_1 + x_2')^2}\}} \quad . \quad . \quad (1.66)$$

In this determination of the maximum torque we developed the relation between $\bar{E}_2$ and $\bar{V}'$, and neglected the effect of the no-load current on the impedance drop. Since $\bar{V}$ appears as a square, any

error produced by neglecting $\bar{I}_0$ will be exaggerated. In all equations for torque and power where $\bar{V}$ occurs, the vector difference of $\bar{V}$ and the primary impedance drop due to the no-load current should appear.

Results, sufficiently approximate, will be obtained by assuming the $\boldsymbol{V} - \boldsymbol{I}_0 x_1$ for $\bar{V}$ in the equations.

We have shown that the torque in synchronous watts = input to the rotor

$$= m_2 \bar{E}_2 \bar{I}_2 \cos \phi_2$$

Now $$\bar{E}_2 = 4{\cdot}44 K_2 K_4 \times T_2 \times \hat{\phi} \times f \times 10^{-8}$$

$$f = \frac{p \times \text{revs per min. (syn.)}}{120}$$

and $$T_2 = \frac{Z_2}{2m_2}$$

where Z_2 = total number of conductors on the rotor

Therefore, torque in synchronous watts

$$= m_2 \times 4{\cdot}44 \times K_2 K_4 \times \frac{Z_2}{2m_2} \times \frac{\hat{\phi} \times p \times \text{r.p.m.}_{\text{syn}} \times \bar{I}_2 \cos \phi_2}{120 \times 10^8} \quad (1.67)$$

The torque in lb-ft

$$= \frac{4{\cdot}44 K_2 K_4 \times Z_2 \times \hat{\phi} \times p \times \bar{I}_2 \cos \phi_2 \times 7{\cdot}04}{240 \times 10^8} \quad . \ (1.68)$$

$$= 0{\cdot}1303 \times 10^{-8} \times p \times \hat{\phi} \times Z_2 \bar{I}_2 \cos \phi_2 \times K_2 K_4 \quad (1.69)$$

This is a very useful relation.

p = number of poles

T_2 = turns per rotor phase in series

$\hat{\phi}$ = maximum flux per pole

Z_2 = total conductors on the rotor

$\bar{I}_2$ = r.m.s. value of rotor current and

$\cos \phi_2$ = cosine of the angle between $\bar{E}_2$ and $\bar{I}_2$.

The torque of a polyphase motor in ft-lb = a constant × total flux × total rotor ampere-conductors × $\cos \phi_2$.

Note that $$\frac{\text{torque in lb-ft} \times \text{syn. r.p.m.}}{7{\cdot}04} = \text{syn. watts.}$$

Therefore, torque in lb-ft × r.p.m.$_{\text{syn}}$

$$= 7{\cdot}04 \times \text{power represented by the syn. watts} \quad . \ (1.70)$$

Equivalent Stator and Rotor Quantities

Let T_1 = turns in series on the stator per phase and

T_2 = turns in series on the rotor per phase.

Then
$$\bar{E}_1 = 4{\cdot}44 \times T_1 \times K_1K_3 \times \hat{\phi} \times f \times 10^{-8} \quad . \quad . \quad (1{\cdot}71)$$
$$\bar{E}_2 = 4{\cdot}44 \times T_2 \times K_2 \times K_4 \times \hat{\phi} \times f \times 10^{-8} \quad . \quad . \quad (1{\cdot}72)$$
$$\frac{\bar{E}_1}{\bar{E}_2} = \frac{T_1 \times K_1 \times K_3}{T_2 \times K_2 \times K_4} = \frac{T_1 f_1}{T_2 \times f_2} \quad . \quad . \quad . \quad . \quad (1{\cdot}73)$$

$f_1 = K_1 \times K_3$ = winding factor for the fundamental wave

= product of distribution factor and coil span factor

f_2 = the same for the rotor.

Now
$$K_1 = \frac{\sin q_1 \frac{\lambda}{2}}{q_1 \sin \frac{\lambda}{2}} \text{ and } K_3 = \cos \frac{\varepsilon}{2} \quad . \quad . \quad (1{\cdot}74)$$

q_1 = number of slots *per pole per phase* in the stator

q_2 = number of slots per pole per phase for the rotor

λ = electrical slot-pitch angle

$$= \frac{180^\circ}{\text{number of slots per pole}} \quad . \quad . \quad . \quad . \quad (1{\cdot}75)$$

λ' = electrical slot pitch angle in rotor

$$f_2 = \frac{\sin q_2 \frac{\lambda'}{2}}{q_2 \sin \frac{\lambda'}{2}} \times \cos \frac{\varepsilon'}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}76)$$

and ε and ε' = *deficiency* from full pitch in electrical degrees, of stator and rotor coils respectively.

Instead of the factor $\cos \frac{\varepsilon}{2}$ we may use $\sin \frac{\beta}{2}$, where β = actual span of the coil in electrical degrees.

For the nth *harmonic*, we have—

$$f_n = \frac{\sin qn \frac{\lambda}{2}}{q \sin n \frac{\lambda}{2}} \times \cos n \frac{\varepsilon}{2} \quad . \quad . \quad . \quad (1{\cdot}77)$$

where f_n = winding factor for the nth harmonic.

Rotor Current Referred to Stator

Since the ampere-turns (effective), due to the load component in the stator must equal the ampere-turns effective of the rotor, we have—

$$m_1 T_1 \bar{I}_1' K_1 K_3 = m_2 T_2 \bar{I}_2 K_2 K_4 \quad . \quad . \quad (1.78)$$

$$\therefore \quad I_1' = \frac{m_2}{m_1} \times \frac{T_2}{T_1} \times \frac{K_2 K_4 \bar{I}_2}{K_1 K_3} = \frac{\bar{I}_2}{\alpha} \quad . \quad . \quad (1.79)$$

where $$\alpha = \text{ratio } \frac{m_1 T_1 K_1 K_3}{m_2 T_2 K_2 K_4} \quad . \quad . \quad . \quad (1.80)$$

The load component of current in the stator is given by equation $(1.79) = \dfrac{\bar{I}_2}{\alpha}$.

Rotor Impedance Referred to Stator

Let Z_2' = equivalent rotor impedance referred to the stator.

$$Z_2' = \frac{\bar{E}_2'}{\bar{I}_1'} = \bar{E}_2 \times \frac{T_1 \times f_1 \times m_1 T_1 \times f_1}{T_2 \times f_2 \times m_2 T_2 \times f_2 \times \bar{I}_2} \quad . \quad (1.81)$$

$$= \frac{\bar{E}_2}{\bar{I}_2} \times \frac{m_1 T_1^2 \times f_1^2}{m_2 T_2^2 \times f_2^2} \quad . \quad . \quad . \quad . \quad . \quad (1.82)$$

$$= Z_2 \times \frac{m_1 T_1^2 \times f_1^2}{m_2 T_2^2 \times f_2^2} \quad . \quad . \quad . \quad . \quad . \quad (1.83)$$

If $m_1 = m_2$, then—

$$Z_2' = Z_2 \times \left(\frac{T_1 \times f_1}{T_2 \times f_2}\right)^2 \quad . \quad . \quad . \quad (1.84)$$

$T_1 \times f_1$ is called the *effective* number of stator turns per phase and $T_2 \times f_2$ the number of *effective* rotor turns per phase.

The resistances and reactances of the rotor are referred to the stator in the same way as the impedances.

Geometrical Relations from the Circle Diagram

Most of the quantities we have dealt with can be read directly from the circle diagram. In Fig. 1.12, the maximum power is proportional to GH, where GH is the largest vertical intercept between the circle and the output line. Draw a line tangential to the circle, and parallel to JF. This line will touch the circle at G. Then the—

$$\text{max. h.p.} = \frac{m_1 \times \bar{V} \times GH}{746} \quad . \quad . \quad . \quad (1.85)$$

$$GH \sin\phi = GM$$

$$GH = \frac{GM}{\sin\phi} = \frac{GP - MP}{\sin\phi} \quad . \quad . \quad (1.86)$$

$$= \frac{GP - JP\cos\phi}{\sin\phi} \quad . \quad . \quad . \quad (1.87)$$

Since $$GP = JP$$

and $$JF = 2GP \sin \phi \qquad (1.88)$$

$$\therefore \quad GP = \frac{JF}{2 \sin \phi} \qquad (1.89)$$

$$\therefore \quad GH = \frac{JF(1 - \cos \phi)}{2 \sin^2 \phi} \qquad (1.90)$$

$$= \frac{JF(1 - \cos \phi)}{2(1 - \cos^2 \phi)} = \frac{JF}{2(1 + \cos \phi)} \qquad (1.91)$$

and $$\text{max. h.p.} = \frac{m \times \bar{V} \times JF}{2(1 + \cos \phi) \times 746} \qquad (1.92)$$

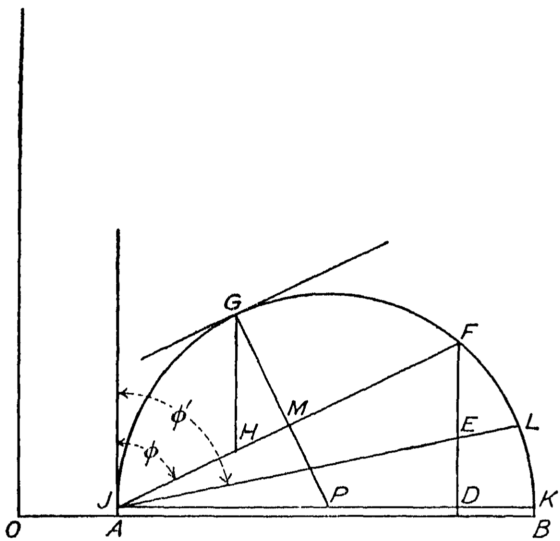


FIG. 1.12

where $\cos \phi =$ the power factor at short-circuit of the current JF and $JF = JK \sin \phi$.

In dealing with the motor, JK, the ideal short-circuit current is usually known, or JF is known from test.

In a similar manner the maximum torque is proportional to the maximum vertical intercept between the circle and the torque line JL.

$$\text{max. torque} = \frac{m_1 V \times JL}{2(1 + \cos \phi')} \qquad (1.93)$$

and $$JL = JK \sin \phi' \qquad (1.94)$$

where ϕ' is the angle JL makes with the vertical.

Torque and Slip Relations

If one neglects the stator copper loss, the diagram takes the simple form shown in Fig. 1.13.

Join DL, where L = standstill point, and produce DL to meet the vertical through M in E.

$\triangle$'s DQM and MOA are similar and so are also the $\triangle$'s DEM and MRA.

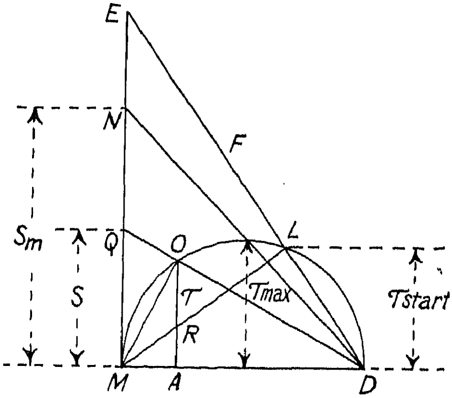


Fig. 1.13

$$\therefore \qquad \frac{RA}{MA} = \frac{MD}{ME} \text{ and } \frac{MQ}{MD} = \frac{MA}{OA} \qquad (1.95)$$

$$\therefore \qquad MQ \times OA = RA \times ME \qquad (1.96)$$

$$\therefore \qquad MQ = \frac{RA}{OA} \times ME \qquad (1.97)$$

but $RA \propto$ loss in the rotor (neglecting stator loss), $OA \propto$ torque.

$$\frac{RA}{OA} = \text{slip} = \frac{MQ}{ME} \qquad (1.98)$$

where ME represents the slip at standstill.

Let $\tau_{\max}$ = maximum torque

s_m = slip at maximum torque

and τ = torque at any other slip $s = OA$.

Then
$$\frac{\tau}{AD} = \frac{MQ}{MD} = \frac{MQ}{MN} = \frac{s}{s_m} \qquad (1.99)$$

$$\frac{MA}{\tau} = \frac{s}{s_m} \text{ or } MA = \tau \times \frac{s}{s_m}$$

$$MD = MA + AD = \tau\left\{\frac{s}{s_m} + \frac{s_m}{s}\right\} \qquad (1.100)$$

$$\tau_{\max} = \tfrac{1}{2}MD$$

$$\therefore \qquad \frac{\tau}{\tau_{\max}} = \frac{2}{\dfrac{s}{s_m} + \dfrac{s_m}{s}} \qquad . \quad . \quad . \quad (1.101)$$

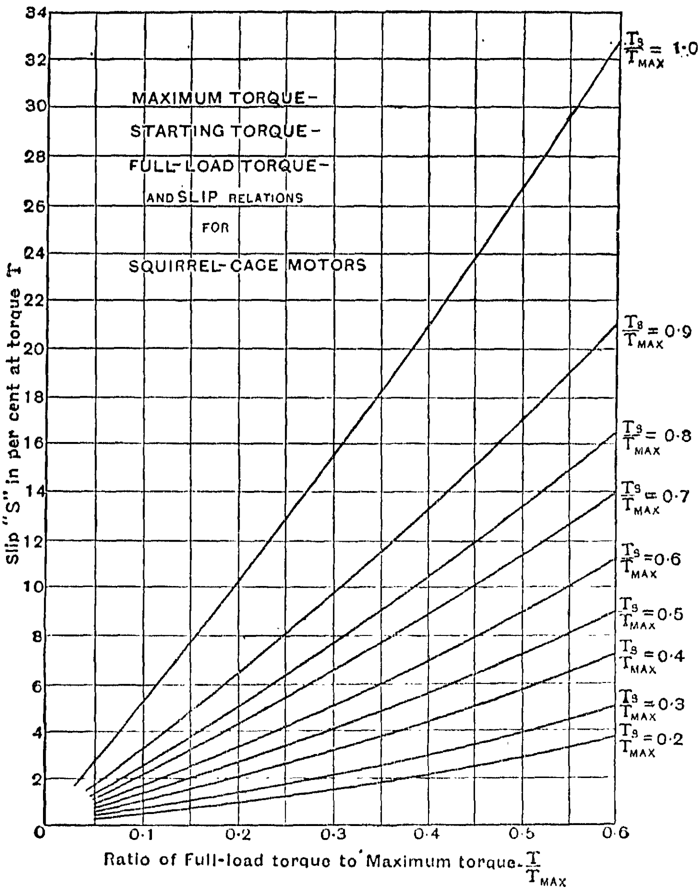


FIG. 1.14

At the starting point $s = 1$

$$\therefore \qquad \frac{\tau_{\text{start}}}{\tau_{\max}} = \frac{2}{\dfrac{1}{s_m} + s_m} \qquad . \quad . \quad . \quad (1.102)$$

These relations are very useful and to show their utility we shall consider an example. A squirrel-cage motor is to be designed for an overload torque capacity of 150 per cent and 20 per cent margin, with a starting torque of 125 per cent of full-load torque. It is required to find the slip at full load.

We have—

$$\frac{\tau}{\tau_{\max}} = 0{\cdot}4 \times 0{\cdot}8 = 0{\cdot}32$$

$$\frac{\tau_{\text{start}}}{\tau_{\max}} = 1{\cdot}25 \times 0{\cdot}32 = 0{\cdot}4$$

From the above equations $s_m = 21$ per cent and $s = 3{\cdot}68$ per cent. Again it is required to find what starting torque can be obtained from a squirrel-cage motor at 100 per cent overload torque and a slip of $2\frac{1}{2}$ per cent at full load.

$$\frac{\tau}{\tau_{\max}} = 0{\cdot}5 \times 0{\cdot}9, \text{ allowing 10 per cent margin}$$

From the equation above, $s_m = 10{\cdot}5$ per cent and

$$\frac{\tau_{\text{start}}}{\tau_{\max}} = 0{\cdot}208$$

i.e.

$$\tau_{\text{start}} = \frac{0{\cdot}208}{0{\cdot}45} \text{ full-load torque}$$

$$= 0{\cdot}46 \text{ of full-load torque}$$

The relationship between slip and torque is shown in Fig. 1.14. It is advisable in using these curves to allow a margin of 10 to 15 per cent on the guaranteed overload torque capacity, on account of the neglect of the stator copper losses and ripples on torque-slip.

The Efficiency

If we ignore all losses, except the rotor copper loss, and since efficiency $= \dfrac{\text{output}}{\text{input}}$ then the output per phase

$$= \bar{E}_2 \bar{I}_2 \cos \phi_2 - \bar{I}_2{}^2 R_2$$

The rotor input per phase

$$= \bar{E}_2 \bar{I}_2 \cos \phi_2$$

Therefore, the efficiency, neglecting all losses but rotor copper losses

$$= 1 - \frac{\bar{I}_2{}^2 R_2}{\bar{E}_2 \bar{I}_2 \cos \phi_2}$$

$$= 1 - s \qquad (1.103)$$

Therefore, the efficiency is always less than the speed as a percentage of synchronous speed, since there are other losses in addition to the rotor copper loss, for

$$1 - s = 1 - \frac{(\omega_0 - \omega)}{\omega_0} = \frac{\omega}{\omega_0} \qquad (1.104)$$

The Dispersion Co-efficient

If we neglect the no-load watt component of current, our circle diagram assumes the simple form of Fig. 1.15.

So long as the voltage is maintained constant at the stator terminals, the flux linked with the stator winding is constant, neglecting resistance drop. With locked rotor the flux is almost entirely leakage flux. At synchronous speed, there is no rotor current and the stator flux is free to enter the rotor. Hence OB represents the current necessary to drive the flux through the leakage

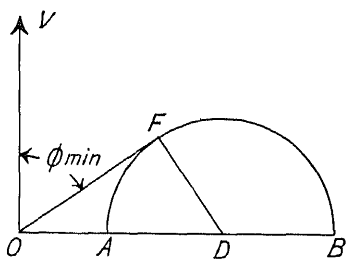


FIG. 1.15

paths, while OA represents the current required to send the same flux through the useful path and the leakage path in parallel.

The leakage factor of any circuit

$$= \frac{\text{total flux}}{\text{useful flux}}$$

$$= \frac{\text{useful flux} + \text{leakage flux}}{\text{useful flux}}$$

$$= 1 + \frac{\text{leakage flux}}{\text{useful flux}} \quad . \quad . \quad . \quad . \quad . \quad (1.105)$$

Now flux $= \dfrac{\text{m.m.f.}}{\text{reluctance}} = \text{m.m.f.} \times \text{permeance}$; and the currents are proportional to the m.m.f.s.

$$\therefore \frac{OB}{OA} = \frac{\text{reluctance of leakage paths}}{\text{joint reluctance of useful paths} + \text{leakage paths}}$$

$$= \frac{\text{permeance of leakage path} + \text{permeance of useful path}}{\text{permeance of leakage path}}$$

$$= 1 + \frac{\text{permeance of useful path}}{\text{permeance of leakage path}}$$

$$= 1 + \frac{\text{useful flux}}{\text{leakage flux}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.106)$$

Also $$\frac{OB}{OA} = \frac{OA + AB}{OA} = 1 + \frac{AB}{OA} \quad . \quad . \quad (1.107)$$

$$\therefore \quad \frac{AB}{OA} = \frac{\text{useful flux}}{\text{leakage flux}}$$

$$\text{The leakage factor} = 1 + \frac{OA}{AB} \quad . \quad . \quad . \quad . \quad (1.108)$$

The symbol "σ" has been used to represent another leakage ratio, called the "dispersion coefficient."

Behn-Eschenberg uses for the dispersion coefficient the ratio $\frac{OA}{OB} = \frac{\text{leakage flux}}{\text{total flux}}$. We shall adopt this definition of the dispersion coefficient, i.e. $\sigma = \frac{\text{magnetizing current per phase}}{\text{ideal short-circuit current per phase}}$.

Hobart and others use another coefficient,

i.e. $$\frac{OA}{AB} = \frac{\text{leakage flux}}{\text{useful flux}} = v$$

$$\begin{aligned}\text{The leakage factor} &= 1 + v \\ &= 1 + \frac{OA}{OB - OA} = 1 + \frac{\sigma}{1 - \sigma} \\ &= \frac{1}{1 - \sigma} \quad . \quad . \quad . \quad . \quad . \quad (1.109)\end{aligned}$$

The value of the dispersion coefficient has a most important influence on the power factor of the motor. If high power factor is needed, then a small value of σ is indicated. That is we must have a small magnetizing current OA, and a large ideal short-circuit current. Now OA is determined principally by the length of air-gap used, for the magnetizing current is proportional to the gap length.

The air-gap length must be kept small, indeed as small as is mechanically permissible, and there must be no saturation in the iron. Thus, one finds in these motors air-gaps of the order of 0·3 mm to 1 mm, depending on the size of motor, and, of course, in very small motors, whose output is but a few watts, the gap length may be as small as 0·006 in.

Large ideal short-circuit current means, of course, small leakage flux reactance, since the ideal short-circuit current $= \frac{\text{flux per pole}}{\text{leakage flux per pole per ampere}}$ it is clear that the dispersion coefficient depends chiefly on the ratio of $\frac{\text{length of air-gap}}{\text{pole pitch}}$.

On a given diameter, and with a given core length, the pole area, and therefore the flux per pole, is dependent on the pole pitch; the flux per pole will be dependent on the pole pitch

As the pole pitch decreases, as the number of poles increases, the number of slots per pole per phase is decreased, and the winding is more concentrated, the number of conductors per pole per phase is increased for a given supply voltage, and hence the leakage reactance is increased. Thus, one must expect both the magnetizing current *OA* to increase and the ideal short-circuit current to decrease as the number of poles in the machine is increased.

Thus, it is natural for high-speed induction motors, i.e. machines with a small number of poles to have small values for the dispersion coefficient. There is thus no difficulty in obtaining high power factor in high-speed motors; but as the number of poles increases, and the speed is low, σ progressively increases, and it becomes difficult to obtain high power factor. Now looking at our diagram, Fig. 1.15, we get maximum power factor when *OF* is tangent to the circle.

Maximum power factor

$$= \cos FOV = \frac{FD}{OD} = \frac{AB}{2 \times OD}$$

$$= \frac{AB}{2OA + AB}$$

$$= \frac{1}{2v + 1}$$

$$= \frac{1}{\dfrac{2\sigma}{1 - \sigma} + 1} = \frac{1 - \sigma}{1 + \sigma} \quad . \quad . \quad . \quad (1.110)$$

where $v = \dfrac{OA}{AB}$

Thus, the maximum power factor—

$$= \frac{1 - \sigma}{1 + \sigma} \quad . \quad . \quad . \quad . \quad (1.111)$$

The influence of σ on the power factor is thus seen to be all-important. In machines of two-, four-, and six-poles, it is easy to obtain power factors of 90 per cent and over, but when σ becomes of the order of 0·1 or more, it is clear that the power factor becomes lower than 90 per cent, indeed, for $\sigma = 0{\cdot}1$, $\cos \phi_{max} = \dfrac{0{\cdot}9}{1{\cdot}1} = 81{\cdot}75$ per cent and the power factor at full load is much lower.

Now air-gaps which are too small, give trouble due to zigzag leakage fluxes and other causes, so the gap should be as large as is compatible with good power factor. These points will receive further consideration later, but it is desirable to point out that unduly short air-gaps are not necessary in high-speed motors, and in some cases, in the author's experience, machines have failed mechanically due to

this desire to obtain unduly large power factors by using very small gap lengths.

The following test results, taken at the Electrical Engineering Laboratory of the University of British Columbia, illustrate the characteristic curves of the three-phase induction motor, and may be helpful to the student. They were taken by one of my former students, Mr. Walter Lind.

THREE-PHASE INDUCTION MOTOR: DIRECT DETERMINATION OF CHARACTERISTICS

Apparatus. Machine No. 3 and No. 17. (C.G.E. induction motor and d.c. generator set.) Ratings—

Induction motor	D.C. generator
60 c/s	10 kW
38·5 A	80 A
220 V	125 V (at no-load and full load)
1800 r.p.m. at no-load	1800 r.p.m.
Exciter voltage 125 V	

Procedure. The d.c. generator was used to load the induction motor. The slip was determined with a stroboscopic disc for light loads.

Observations—

Induction Motor										D.C. Generator			
Volts		Amperes		Watts					Slip (r.p.m.)	Term. Volts	Arm. Amperes I_a	Field Amperes I_{SH}	Watts Output
ϕ_1	ϕ_3	ϕ_1	ϕ_3	W_1	W_2	$W_1 + W_2$ (Input)	W_2/W_1	P.F.					
231	231	11·3	12·0	1590	− 680	910	− 0·427	0·23	3·5	—	0·0	—	—
230	232	11·5	12·2	1850	− 450	1 400	− 0·243	0·33	6·2	120·5	0·0	1·69	20
229	229	14·6	15·6	3120	775	3 895	0·248	0·69	22·0	120·5	20·0	1·73	2 64
224	226	20·2	22·0	4450	2200	6 650	0·495	0·87	40·0	121·0	41·1	1·74	5 29
226	227	26·8	29·3	5980	3450	9 430	0·577	0·91	60·0	121·5	59·9	1·75	7 71
226	226	35·7	38·0	7900	4950	12 850	0·626	0·93	80·0	121·0	81·6	1·75	10 50
225	227	43·7	46·2	9580	6350	15 930	0·663	0·94	110·0	121·5	98·5	1·82	12 78

Resistance Measurements (after the load test)—

Armature			Series Field			Interpoles			Total $R_a + R_s + R_i$ ohms
Volts	Amperes	R_a ohms	V	I_s	R_s ohms	V	I_i	R_i ohms	
0·55	16·0	0·0344	0·24	17·5	0·0137	0·41	28·3	0·0145	Mean value 0·0618
1·24	37·0	0·0336	0·46	35·0	0·0131	0·50	33·4	0·0149	
1·11	33·0	0·0336	0·45	33·0	0·0136	—	—	—	

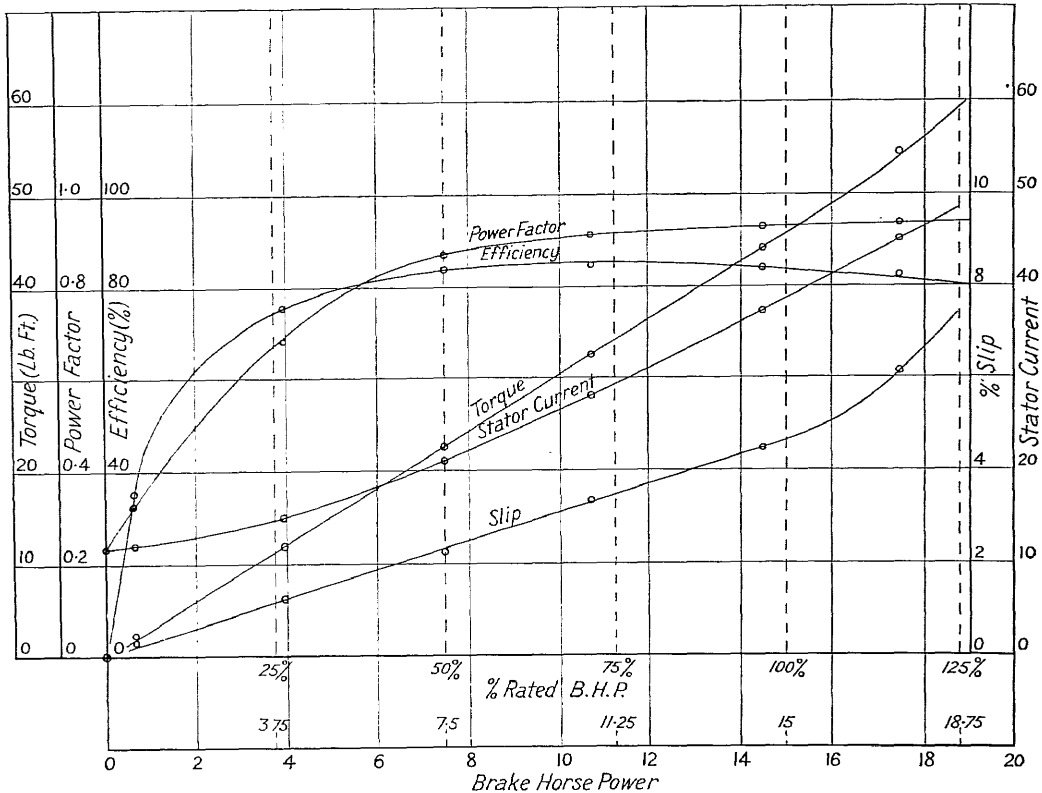


FIG. 1.16. DIRECT DETERMINATION OF THE CHARACTERISTIC OF A THREE-PHASE INDUCTION MOTOR

Calculations (see also Fig. 1.16)

Motor	Generator							M
Input watts	VI_a watts	Copper Loss watts	Core Loss watts	Total Output watts	Torque ft-lb	Output h.p.	Slip %	E
910	0	0	0	0	0	0	0·19	
1 400	0	203	287	490	1·9	0·66	0·35	3
3 895	2 410	234	—	2 930	11·6	3·93	1·22	7
6 650	4 980	315	—	5 580	22·4	7·48	2·22	8
9 430	7 280	434	—	8·000	32·4	10·70	3·34	8
12 850	9 880	625	—	10 790	44·2	14·50	4·45	8
15 930	11 960	820	—	13 070	54·5	17·50	6·12	8

Case 3

$$\text{Copper losses} = I_a^2R + VI_f = (20)^2(0{\cdot}0618) + (120{\cdot}5)(1{\cdot}73$$

$$= 24{\cdot}7 + 209 \qquad \simeq 234$$

$$\text{Core loss} = 1400 - 910 - 203 \qquad = 287$$

$$\text{Generator output} = VI = (120{\cdot}5)(20) \qquad = 2410$$

$$\text{Motor output} = 2410 + 234 + 287 \qquad = 2930$$

$$\text{Torque} = \frac{(33\,000)(\text{watts output})}{(2\pi N)(746)} = \frac{(7{\cdot}05)(\text{output})}{N}$$

Full-load current = 38·5 A

Approximate full-load output (gross) $= \sqrt{3}VI$

$$= \sqrt{3}(220)(38{\cdot}5) = 14{\cdot}7 \text{ k}$$

$$= 19{\cdot}7 \text{ h}$$

Assume p.f. = 0·92 at full load

Assume efficiency = 0·83 at full load

Then the net full-load output = (19·7)(0·92)(0·83) = 15 h.p

THE CIRCLE DIAGRAM OF THE INDUCTION MOTOR

Apparatus. As in previous experiment.

Procedure—

No-load Test

Volts		Amperes		Watts		$\frac{W_1}{W_2}$	P.F. Cos ϕ	θ_{NL}
1	2	1	2	1	2			
227	229	11·15	11·85	1750	− 760	− 0·434	0·225	77°

Locked Rotor Test

Volts		Amperes		Watts		$\frac{W_1}{W_2}$	P.F. Cos ϕ	θ_s
1	2	1	2	1	2			
25·8	27·8	23·0	22·5	565	100	0·177	0·630	50° 00′
32·3	34·0	30·0	27·0	880	90	0·120	0·590	53° 00′
39·0	40·7	35·0	32·8	1270	200	0·157	0·610	52° 30′
45·0	47·0	40·0	38·0	1584	240	0·152	0·610	52° 30′
50·8	53·6	45·0	43·0	2040	360	0·176	0·630	50° 00′
58·0	58·2	48·4	49·3	2500	510	0·204	0·650	49° 30′
63·5	63·0	52·2	53·0	3020	688	0·228	0·675	47° 30′
							Average	50° 50′

Stator Resistance

Volts	Amps.	$2R$	R
2·34	8·35	0·28	0·14
5·70	20·50	0·28	0·14

Average resistance per phase of stator $= 0{\cdot}14\ \Omega$

Calculations (see also Figs. 1.17, 1.18, 1.19)

Short-circuit current at operating voltage of previous experiment

$$= \left(\frac{50}{59{\cdot}6}\right)(228) \doteqdot 190\ \text{A}$$

Full-load h.p. (see previous experiment) $= 15$ h.p.

Full-load h.p. per phase $= 5$ h.p.

Increase in stator copper loss per phase

$$= R(I_s^2 - I_N^2) = (0{\cdot}14)(190^2 - 11{\cdot}5^2) \doteqdot 5{\cdot}0\ \text{kW}$$

$$\doteqdot 2{\cdot}54\ \text{kW}$$

Full Load %	θ	P.F. Cos θ	Stator Amperes	Slip %	Torque 3 phase lb-ft	Efficiency %
25	46° 00′	0·70	15·5	0	11·0	62
50	33° 00′	0·84	22·0	3·0	22·5	78
75	26° 00′	0·90	30·0	5·8	35·0	83
100	23° 00′	0·92	38·4	7·7	47·5	81
125	22° 00′	0·93	49·0	11·0	61·0	78
150	22° 30′	0·92	62·0	14·5	77·0	75

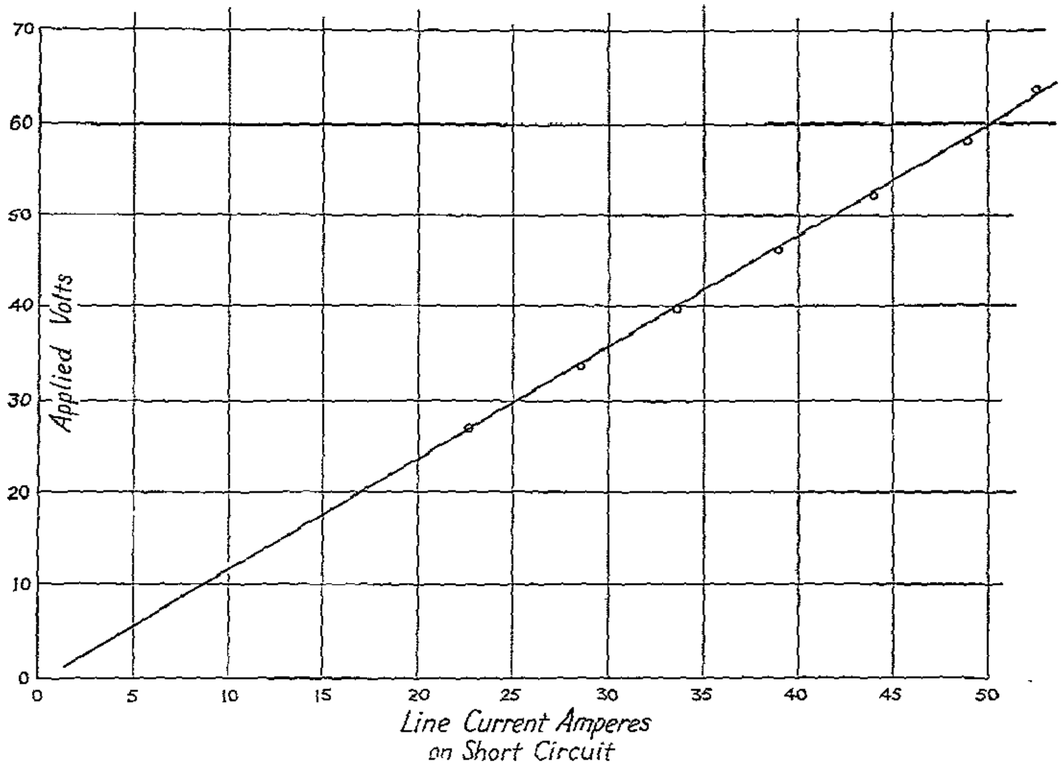


Fig. 117. Short-circuit Characteristic for a Three-phase Induction Motor

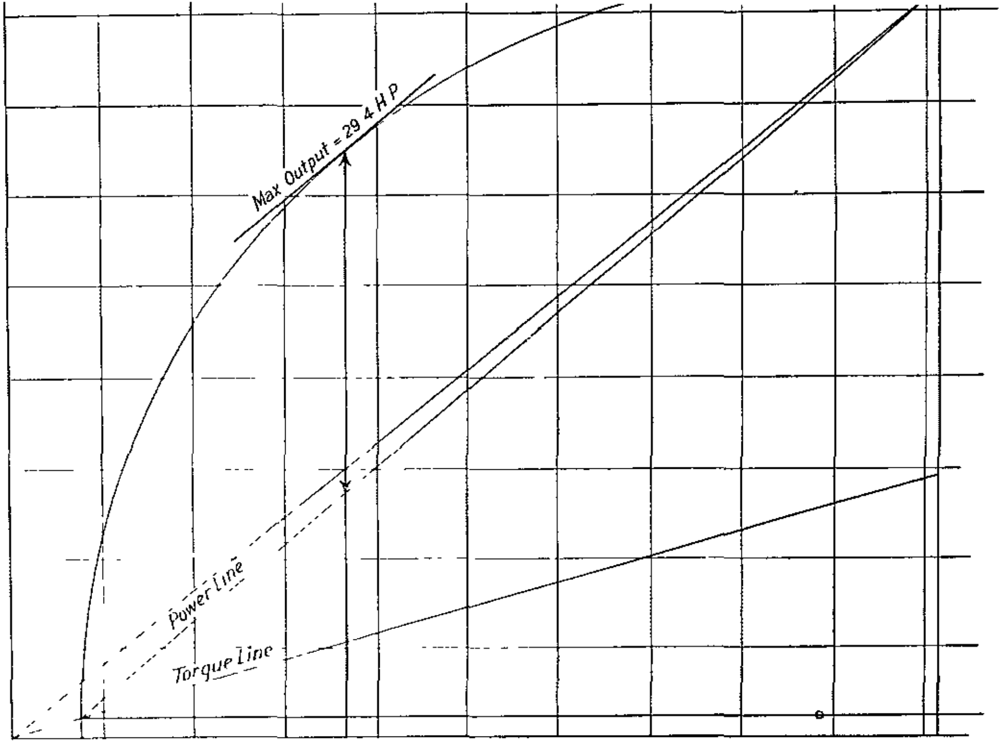
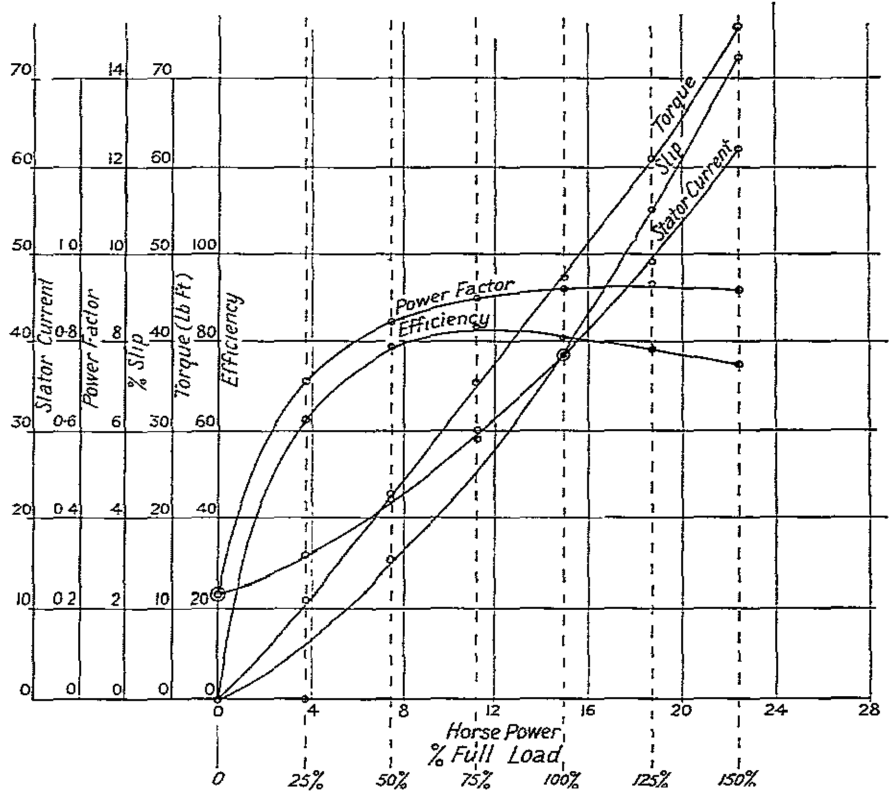


Fig. 118 The Circle Diagram of the Induction Motor



Torque
Slip
Stator Current
Power Factor
Efficiency
Stator Current
Power Factor
% Slip
Torque (Lb Ft)
Efficiency
0 10 20 30 40 50 60 70
0 0.2 0.4 0.6 0.8 1.0
0 2 4 6 8 10 12 14
0 10 20 30 40 50 60 70
0 20 40 60 80 100
0 4 8 12 16 20 24 28
Horse Power
% Full Load
0 25% 50% 75% 100% 125% 150%

THREE-PHASE SLIP-RING INDUCTION MOTOR

Test Equipment. Crompton and Parkinson Limited, slip-ring duction motor.

b.h.p. = 10	1135 r.p.m.
Volts = 220	Mesh = 29 A
C/s = 60	Rotor = 33 A

Observations—

Locked Rotor Test

Volts		Amperes		Watts		P.F. Cos ϕ
1	2	1	2	1	2	
34·8	35·3	14·9	15·0	370	− 100	0·32
44·0	43·0	20·2	19·8	630	− 175	0·31
61·5	61·0	30·5	30·0	1420	− 330	0·34
70·0	69·0	35·0	34·5	1760	− 375	0·35
79·0	78·0	40·3	40·0	2500	− 410	0·38
88·0	86·5	45·3	44·5	2970	− 620	0·35
					Mean P.F. 0·34	

No-load Test

Volts		Amperes		Watts		P.F. Cos ϕ
1	2	1	2	1	2	
229	230	14·6	13·8	1920	− 1280	0·11
229	230	14·8	14·0	1925	− 1305	0·11

Stator : Rotor Ratio Test

Three-phase			Single-phase		
Stator Volts	Rotor Volts	Ratio	Stator Volts	Rotor Volts	Ratio
227	131·0	1·69	228	136	1·68
228	134·5	1·69	228	135	1·69

...ance of stator per phase $= \dfrac{0{\cdot}3}{2} = 0{\cdot}15\ \Omega$
...sistance of rotor per phase $= \dfrac{0{\cdot}18}{2} = 0{\cdot}09\ \Omega$ } Using the duct...

Maximum resistance of starter per phase $= 4{\cdot}78\ \Omega$ Volt/Amp meth...

Starting Torque Test

Volts		Amperes		Watts		Weight lb	Scale lb	Torq... lb-ft
1	2	1	2	1	2			
228	228	18·5	18·0	425	3800	11	6·5	21·8

Effective weight of Prony brake arm = 4·2 lb
Effective length of Prony brake arm = 30 in.

Calculations (see also Figs. 1.20, 1.21)

Starting torque $= (W - s)(\text{lever arm}) = (11 + 4{\cdot}2 - 6{\cdot}5)(2{\cdot}5)$
$= 21{\cdot}8$ lb-...

Short-circuit stator current at working voltage 228

$$= \left(\frac{228}{87 - 6}\right)(45) = 125\ \text{A}$$

(This assumes that the voltmeter had a constant error of 6 V, to permit graph (1) to pass through the origin.)

P.F. at no-load $= 0{\cdot}11$ and $\therefore\ \theta_N = 83^\circ\ 30'$
Mean S.C.–P.F. $= 0{\cdot}34$ and $\therefore\ \theta_s = 70^\circ$

Increase in stator copper loss from no-load to locked rotor condition at 228 V $= R_1(I_s^2 - I_N^2)$
$= (0{\cdot}15)(125^2 - 14{\cdot}3^2) \quad = 2310$ W

From Diagram—

Full-load rotor current $= 34{\cdot}7$ A
Full-load starting torque $= (bd)(\text{scale})$
$\therefore$ Rotor resistance for full-load starting torque

$$= \frac{(bd)(658)}{I_R^2} = 2{\cdot}1\ \Omega$$

and the extra rotor resistance required is, therefore,

$$= 2{\cdot}1 - 0{\cdot}09 \doteqdot 2\ \Omega$$

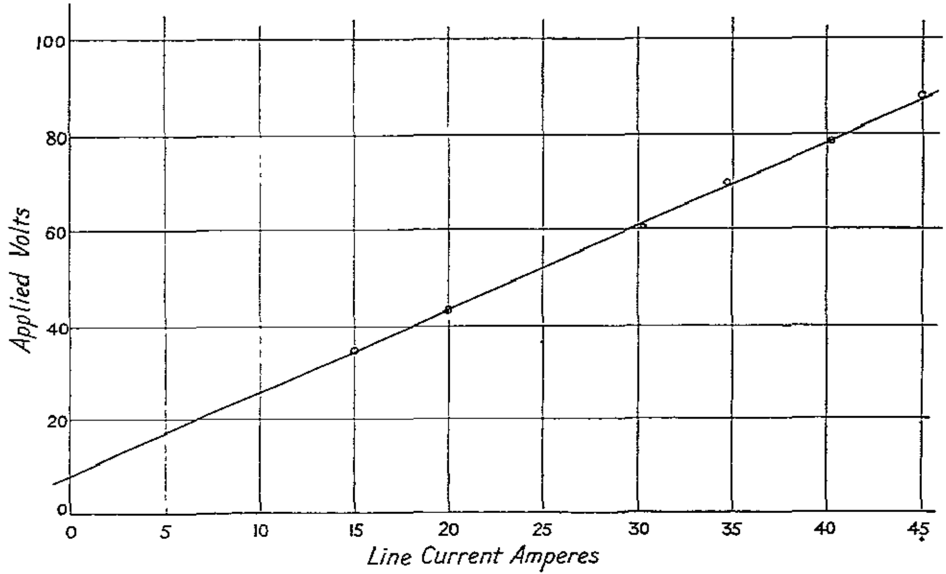


Fig. 3.20 Locked-rotor Test for Three-phase Slip-ring Induction Motor

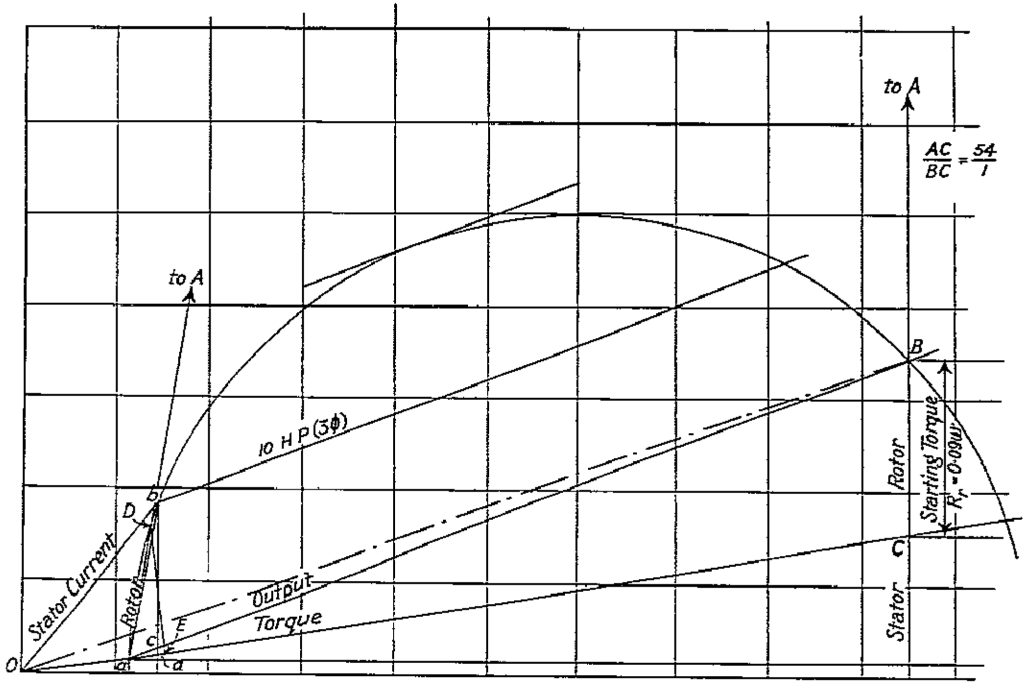


FIG 1.21. THREE-PHASE SLIP-RING INDUCTION MOTOR

'o determine the starting torque and stator and rotor current 1 the resistance of the rotor circuit is $(4{\cdot}78\omega + 0{\cdot}09\omega)$ erect the cal line ABC, so that $\dfrac{AC}{BC} = \dfrac{4{\cdot}78 + 0{\cdot}09}{0{\cdot}09} \simeq 54 : 1$—join A to a.

$\begin{cases}\text{Then stator current} = OD & = 18{\cdot}3\text{ A}\\ \text{Measured (observed) stator current} & = 18{\cdot}3\text{ A}\end{cases}$

Rotor current $= 14{\cdot}8$ A

Starting torque $\simeq 6{\cdot}76$ lb-ft per phase

$\begin{cases}\text{Starting torque (three-phase)} & \simeq 20{\cdot}3\text{ lb-ft}\\ \text{Observed starting torque} & = 21{\cdot}8\text{ lb-ft}\end{cases}$

SUMMARY OF RESULTS

useful to have all the important results collected together eference. For this purpose reference will be made to Fig. 1.7 e 18).

ı) The magnetizing current per phase $= O'C$

) The ideal short-circuit current per phase $= O'L$

) The no-load current per phase $= O'O$

l) The actual short-circuit current per phase $= O'P$

) The dispersion co-efficient $= \dfrac{\text{magnetizing current}}{\text{ideal short-circuit current}}$

$$= \frac{O'C}{O'L} = \sigma$$

f) Maximum power factor $= \dfrac{1 - \sigma}{1 + \sigma}$

;) OP is the output line, and with stator current $O'B = I_1$ the ut in watts $= m_1\bar{V} \times BD$

e m_1 = number of stator phases

$\bar{V}$ = applied volts per phase (r.m.s. value)

s the vertical intercept from B to the output line OP.

ı) The maximum output in watts $= m_1 \times \bar{V} \times TW$

'he point T is found by drawing a line parallel to OP tangent to ircle and TW is the vertical intercept between T and the line OP.

$$\text{Maximum output in horse power} = \frac{m_1\bar{V} \times TW}{746}$$

) The output with stator current $O'B$ in h.p. $= \dfrac{m_1\bar{V} \times BD}{746}$

j) The loss in the rotor, with load corresponding to $O'B$,

$$= m_1\bar{V} \times DE \text{ watts}$$

(*k*) The loss in the stator, with current $O'B = m_1\bar{V} \times EF$

(*l*) The no-load losses, namely iron, friction, and windag no-load copper loss $= m_1\bar{V} \times FK$

(*m*) The input $= m_1\bar{V} \times BK$.

(*n*) The no-load power factor $= \cos OO'V$.

(*o*) The full-load power factor $= \cos BO'V$.

(*p*) The slip $= \dfrac{\text{rotor copper loss}}{\text{rotor input}}$

$$= \frac{\text{rotor copper loss}}{\text{torque} \times \text{synchronous speed in watts}}$$

$$= \frac{DE}{BE} = \frac{CE}{QE} \text{ in Fig. 1.10.}$$

(*q*) The torque multiplied by the synchronous speed, expr in watts = input to the rotor circuit $= m_2 I_2{}^2 \dfrac{R_2}{s}$

where m_2 = number of rotor phases

I_2 = rotor current

R_2 = rotor resistance per phase, and

s = slip.

The torque in *lb-ft* (gross)

$= 0{\cdot}1303 \times 10^{-8} \times p\hat{\phi} \times Z_2 I_2 \times \cos\phi_2 \times K_2 \times K_3$

$= 0{\cdot}1303 \times 10^{-8} \times$ flux per pole (max.)

$\times$ number of poles

$\times$ *total* number of rotor conductors

$\times$ rotor current per conductor

$\times$ power factor of rotor current

$\times$ breadth factor of rotor winding

$\times$ coil span factor of rotor winding

(*r*) The maximum torque in synchronous watts $= m_1\bar{V} \times V$ Fig. 1.7

where VX = maximum vertical intercept between the circle and line OG

(*s*) At P, the standstill point, the slip = 1, and

$m_1\bar{V} \times PG$ = rotor copper loss

$m_1\bar{V} \times GH$ = stator copper loss with current OP

(*t*) Slip $= \dfrac{\text{synchronous speed} - \text{actual speed}}{\text{synchronous speed}} = \dfrac{\omega_0 - \omega}{\omega_0}$

actual speed $= \omega_s(1 - s)$

(a) OB = rotor current referred to stator $= I_1'$

$$= \frac{\text{actual rotor current}}{\text{ratio of transformation}} = \frac{I_2}{\alpha}$$

(b) The ratio of transformation $= \dfrac{T_1 \times K_1 \times K_3}{T_2 \times K_2 \times K_4} = \alpha.$

e T_1 = number of turns in series per phase in the stator

T_2 = number of turns per phase in rotor in series

K_1 = breadth factor of the stator winding for the fundamental wave

$$= \frac{\sin q_1 \frac{\lambda}{2}}{q_1 \sin \frac{\lambda}{2}}$$

where q_1 = number of stator slots per pole per phase

λ = electrical slot-pitch angle

$$= \frac{180}{\text{number of slots per pole}}$$

K_3 = coil span factor for the fundamental

$$= \cos \frac{\varepsilon}{2} \text{ or } \sin \frac{\beta}{2}$$

where ε = deficiency, from full span, of the coil in electrical degrees

β = actual span in electrical degrees

K_2 = breadth factor for the fundamental of the rotor winding

$$= \frac{\sin q_2 \frac{\lambda_2}{2}}{q_2 \sin \frac{\lambda_2}{2}}$$

$$K_4 = \cos \frac{\varepsilon_2}{2} \text{ or } \sin \frac{\beta_2}{2}$$

or the nth harmonic, $K_{1n} = \dfrac{\sin nq_1 \frac{\lambda}{2}}{q_1 \sin n \frac{\lambda}{2}}$

$$K_{3n} = \cos n \frac{\varepsilon}{2}$$

If the number of phases in the stator is not the same as in rotor, then—

$$I_1' = \frac{m_2 T_2 \times K_2 \times K_4}{m_1 T_1 \times K_1 \times K_3} \times I_2$$

(*w*) The rotor resistance, *referred to the stator*

$$= \frac{m_1 T_1^2 \times f_1^2 R_2}{m_2 T_2^2 \times f_2^2}$$

where R_2 = rotor resistance per phase

$f_1 = K_1 \times K_3$

$f_2 = K_2 \times K_4$

The rotor reactance per phase, at supply frequency, *referred to the stator*

$$= X_2 \times \frac{m_1 T_1^2 \times f_1^2}{m_2 T_2^2 \times f_2^2}$$

where $T_1 f_1$ = number of *effective* turns per stator phase

$T_2 f_2$ = number of *effective* turns per rotor phase

(*x*) The slip for maximum torque

$$= s_m$$

$$= \pm \frac{R_2}{X_2}$$

$$= \pm \frac{\text{rotor resistance per phase}}{\text{rotor reactance per phase at standstill}}$$

(*y*) Maximum torque $= \dfrac{mE_2^2}{2X_2}$

(*z*) Slip for maximum output $= -\dfrac{R_2^2}{X_2^2} \pm \dfrac{R_2}{X_2}\sqrt{\dfrac{R_2^2}{X_2^2} + 1}$

$$\text{Maximum output} = \frac{m_2 E_2^2}{2X_2^2}(Z_2 - R_2)$$

where $Z_2 = \sqrt{R_2^2 + X_2^2}$

CHAPTER II

The Squirrel-cage Rotor

he induction machine, using a squirrel-cage winding, we have a nber of copper bars, of circular or rectangular section, joined ether at each end by a ring. If we assume the flux to be sinu- ally distributed and rotating relatively to the rotor, it is clear that .f.s will be generated in each bar, the magnitude of each e.m.f. ng determined, at any instant, by the magnitude of the flux sity cutting the bar. If the slip is very small, the rents will be practically in phase with the gener- l e.m.f.s; the distribution of current will also be ısoidal; in other words, the curve of flux dis- ution will coincide with the curve of current ribution, each being a sine curve in space.

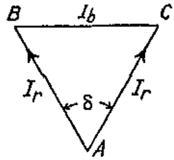


FIG. 2.1

Under the north poles, the current will flow in direction; under the south poles, the current flow in the opposite direction. Thus the rotor currents will pro- e fluxes with as many north and south poles as those produced the stator winding. Thus, the first thing to notice is that the irrel-cage rotor adapts itself to any number of stator poles, and, ourse, it is a condition for constant torque that this should be so. Secondly, we notice that, since the flux is distributed sinusoidally confine our attention to the fundamental wave of flux), there is onstant phase difference between the e.m.f.s and currents in secutive bars, and this phase difference is equal to the electrical pitch angle. This angle of phase difference

$$= \frac{p \times 180}{Q_2}$$

ere p = poles

Q_2 = number of bars

If $\quad I_b$ = current per bar (R.M.S. value) . . (2.1)

$\quad I_r$ = current per ring . . . (2.2)

Then $$I_b = 2I_r \sin\frac{\delta}{2} \quad . \quad . \quad . \quad . \quad (2.3)$$

ere $\delta = \frac{\pi p}{Q_2}$, in electrical radians

If AB and AC represent the currents in consecutive ring secti then the current per bar

$$= 2AB \sin\frac{\delta}{2} \quad . \quad . \quad . \quad . \quad ($$

$$I_b = 2I_r \sin\frac{\delta}{2} \quad . \quad . \quad . \quad . \quad ($$

and
$$I_r = \frac{I_b}{2\sin\frac{\delta}{2}} = \frac{I_b}{2\sin\frac{\pi p}{2Q_2}} \quad . \quad . \quad . \quad ($$

If Q_2 is large, then $\sin\frac{\delta}{2}$ is nearly equal to $\frac{\delta}{2}$, and

$$I_r = \frac{I_b}{\frac{\pi p}{Q_2}} \quad . \quad . \quad . \quad . \quad (2$$

The maximum current in the ring is equal to the average curr per bar, namely, $\frac{2}{\pi}\hat{I}_b \times$ half the bars per pole.

$$\therefore \qquad \hat{I}_r = \frac{2}{\pi}\hat{I}_b \times \frac{Q_2}{2p} = \frac{\hat{I}_b}{\frac{\pi p}{Q_2}} \quad . \quad . \quad (2$$

where $\hat{I}_r$ = maximum value of I_r
and $\hat{I}_b$ = maximum value of I_b

$$\therefore \qquad I_r = \frac{I_b}{\frac{\pi p}{Q_2}} \text{ (R.M.S. values)} \quad . \quad . \quad (2.$$

The copper losses in the squirrel cage

$$= Q_2(I_b{}^2 r_b + 2I_r{}^2 r_r) \quad . \quad . \quad . \quad (2.1$$

where r_b = resistance *per bar*
and r_r = resistance of a single segment of the ring, i.e. the resistance of the part of the ring between two bars.

Therefore, the copper loss in the squirrel-cage rotor

$$= Q_2\left[I_b{}^2 r_b + \frac{2I_b{}^2 r_r}{4\sin^2\frac{\pi p}{2Q_2}}\right] \quad . \quad . \quad (2.1$$

$$= Q_2 I_b{}^2\left[r_b + \frac{r_r}{2\sin^2\frac{\pi p}{2Q_2}}\right] \quad . \quad . \quad . \quad (2.1$$

$$= Q_2 I_b{}^2[r_b'] \quad . \quad . \quad . \quad . \quad . \quad (2.1$$

re
$$r_b' = r_b + \frac{r_r}{2\sin^2\dfrac{\pi p}{2Q_2}} \quad . \quad . \quad . \quad . \quad (2.14)$$

r_b' = resistance of an equivalent bar, which gives the same loss as the actual losses in the bar and its associated segments

If the resistance of each end ring is R_r, then

$$r_r = \frac{R_r}{Q_2} \quad . \quad . \quad . \quad . \quad (2.15)$$

The copper loss in the rotor

$$= I_b^2\left[Q_2 r_b + \frac{Q_2 r_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)}\right] \quad . \quad . \quad (2.16)$$

R_b = resistance of *all* the bars $= Q_2 r_b$
R_r = resistance of *one* ring $= Q_2 r_r$

Then the copper loss in the rotor

$$= I_b^2\left[R_b + \frac{R_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)}\right] \quad . \quad . \quad . \quad (2.17)$$

The squirrel-cage is a polyphase winding, and the number of phases

$$= m_2 = \frac{2Q_2}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.18)$$

= number of bars *per pair* of poles

Consider now the reactance of the rotor and the reactance of an ivalent bar, i.e. a bar with a reactance equal to that of one rotor plus the reactance of the associated segments. Apply Kirchhoff's s to the currents at a junction of a bar and the ring, and also Law 2 ne volts drop around a loop of two bars and two ring segments.

We have
$$I_{b2} = I_{r2} - I_{r1} \quad . \quad . \quad . \quad . \quad (2.19)$$

$$I_{b2} = 2I_R \sin\frac{\delta}{2} \quad . \quad . \quad . \quad . \quad (2.20)$$

have
$$2I_b \sin\frac{\delta}{2} Z_b + \frac{2I_b Z_r}{2\sin\dfrac{\delta}{2}} = 2E_b \sin\frac{\delta}{2} \quad . \quad . \quad (2.21)$$

re Z_b = impedance per bar
Z_r = impedance of one ring sector between two bars

$$I_b = \frac{E_b}{Z'} = \frac{E_b}{Z_b + \dfrac{Z_r}{2\sin^2\dfrac{\delta}{2}}} = \frac{E_b}{Z_b + \dfrac{Z_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)}} \quad . \quad (2.22)$$

and the equivalent impedance per bar

$$= Z' = Z_b + \frac{Z_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)} \qquad . \quad . \quad . \quad (2.2$$

We have already shown that

$$r' = r_b + \frac{r_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)} \qquad . \quad . \quad . \quad (2.2$$

so the equivalent reactance per bar

$$= x_b' = x_b + \frac{x_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)} \qquad . \quad . \quad . \quad (2.2$$

where x_b = reactance per bar

x_r = reactance of one ring segment between two bars

E_b = e.m.f. generated in the bar by the slip of the field.

The copper losses in the squirrel-cage rotor

$$= Q_2 I_b^2 \left(r_b + \frac{r_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)}\right) = Q_2 I_2^2 (r_b') \qquad . \quad (2.26$$

for $\quad I_B = I_2 \quad$ and $\quad r_b' = r_b + \dfrac{r_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)}$ . . (2.27

Now $\quad Q_2 I_2^2 r_b' = m_1 \times (I_1')^2 \times R_2'$ . . (2.28

where R_2' is chosen to represent a resistance which will give the sam loss in the stator as occurs in the rotor per phase

and $\quad m_1$ = number of stator phases

$$\therefore \qquad \frac{I_2^2}{(I_1')^2} = \frac{m_1 R_2'}{Q_2 r_b'} \qquad . \quad . \quad . \quad . \quad (2.29$$

Also $\quad Q_2 I_2 = m_1 \times 2T_1 \times K_1 K_3 \times I_1'$ . . (2.30)

$$\therefore \qquad R_2' = \frac{Q_2 r_b'}{m_1} \times \frac{I_2^2}{I_1'^2} \qquad . \quad . \quad . \quad . \quad . \quad (2.31)$$

$$= \frac{Q_2 r_b'}{m_1} \times \frac{4T_1^2 \times K_1^2 \times K_3^2 \times m_1^2}{Q_2^2} \qquad . \quad (2.32)$$

$$= r_b' \times \frac{4m_1 \times T_1^2 \times K_1^2 \times K_3^2}{Q_2} \qquad . \quad . \quad (2.33)$$

Thus, the equivalent resistance of the rotor referred to the stator

$$= r_b' \times \frac{4m_1 T_1^2 \times K_1^2 \times K_3^2}{Q_2} \qquad . \quad . \quad (2.34)$$

here $r_b' = r_b + \dfrac{r_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)}$

r_b = resistance of one bar

$r_r = \dfrac{\text{resistance of one ring}}{Q_2}$

m_1 = number of phases in the stator

d Q_2 = number of rotor bars or slots

In the same way

$$X_2' = x_b' \times \frac{4m_1 T_1^2 \times K_1^2 \times K_3^2}{Q_2} \quad . \quad . \quad (2.35)$$

here X_2' = reactance of the cage referred to the stator per phase at the supply frequency

d $x_b' = x_b + \dfrac{x_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)}$

where x_b = reactance of *one* bar at the supply frequency

and $x_r = \dfrac{\text{reactance of one ring at supply frequency}}{Q_2}$

If the rotor slots are skewed, and this is done to reduce the effects the harmonics in the flux wave, then in referring the rotor resisnce and reactance to the stator, we must introduce the skew factor to equations (2.34) and (2.35).

Let K_5 = skew factor, which

$$= \sin\frac{\dfrac{C}{\tau_2} \times \dfrac{\alpha}{2}}{\dfrac{C}{\tau_2} \times \dfrac{\alpha}{2}} \quad . \quad . \quad . \quad . \quad (2.36)$$

here C = amount of skew

= BC (*see* Fig. 2.2)

τ_2 = rotor slot pitch

d $\alpha = \dfrac{p\pi}{Q_2}$

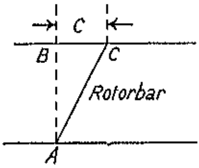


Fig. 2.2

Then for skewed rotor slots, we have

$$R_2' = r_b' \times \frac{4m_1 T_1^2 \times K_1^2 \times K_3^2}{Q_2 \times K_5^2} \quad . \quad . \quad (2.37)$$

d

$$X_2' = x_b' \times \frac{4m_1 T_1^2 \times K_1^2 \times K_3^2}{Q_2 \times K_5^2} \quad . \quad . \quad (2.38)$$

It will also be noticed that the squirrel-cage can be considerec a winding with m_2 phases, where $m_2 = \frac{Q_2 \times 2}{p}$

The Double Cage

This was introduced by Dobrowolski. In this arrangement, t separate cage windings are used; one, the outer cage, has bar smaller cross-section than the inner cage. The outer cage, i.e. one nearer the gap, has relatively high resistance and low leak inductance; the inner cage has conductors of larger cross-sectio usually narrow, deep bars, and it has low resistance and h leakage inductance.

At the start, the frequency of the rotor currents is equal to supply frequency, and the currents in the inner cage are reduced the large impedance of this ca The currents in the outer ca are very effective in produc torque at the start, for the los in this cage are large.

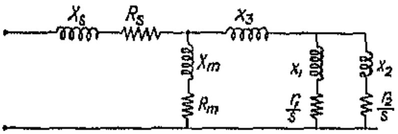


Fig. 2.3

The outer cage is, therefo most effective as regards starti torque. With rising speed, the slip frequency of the rotor curr decreases, and the inner cage becomes effective, since its reacta becomes smaller, and its current increases. At full load the reactan are negligibly small compared to the resistances, and the currents each cage are determined by the resistances almost entirely. Th the starting torque is increased, with smaller starting curre than in a single-cage motor, and some of the advantages of t slip-ring type machine are obtained. The inner bar is separat from the outer bar by a fairly long narrow slit, and the outer b is usually of circular section, and the inner bar of rectangu section—narrow and deep.

Let r_1 = resistance of inner cage, referred to the stator

x_1 = leakage reactance, at supply frequency, of the in cage, referred to the stator

r_2 = resistance of outer cage, referred to the stator

x_2 = leakage reactance of outer cage, referred to the stat at supply frequency

x_3 = leakage reactance, referred to the stator, at sup frequency, due to leakage flux linking both cages

Then the equivalent circuit of the double-cage motor is as Fig. 2.3.

$\boldsymbol{Z}_1$ = impedance of *inner* cage, referred to the stator

$$= \frac{r_1}{s} \times jx_1 \quad . \quad . \quad . \quad . \quad . \quad (2.3$$

$\mathbf{Z}_2$ = impedance of outer cage, referred to the stator

$$= \frac{r_2}{s} + jx_2 \quad . \quad . \quad . \quad . \quad (2.40)$$

The combined impedance

$$= \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.41)$$

$$= \frac{\dfrac{r_1r_2}{s^2} + j\left(\dfrac{x_1r_2}{s} + \dfrac{x_2r_1}{s}\right) - x_1x_2}{\dfrac{r_1 + r_2}{s} + j(x_1 + x_2)} \quad . \quad . \quad (2.42)$$

Since x_2 is usually negligible

$$\mathbf{Z} \simeq \frac{\dfrac{r_1r_2}{s^2} + \dfrac{jr_2x_1}{s}}{\dfrac{r_1 + r_2}{s} + jx_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.43)$$

$$\mathbf{Z} \simeq \frac{\dfrac{r_1r_2}{s} + jr_2x_1}{r_1 + r_2 + jsx_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.44)$$

$$\simeq \frac{r_1r_2 + jsr_2x_1}{s\{(r_1 + r_2) \times jsx_1\}}$$

$$\simeq \frac{(r_1r_2)(r_1 + r_2) + s^2r_2x_1^{\,2} + j\{sr_2x_1(r_1 + r_2) - sx_1r_1r_2\}}{s\{(r_1 + r_2)^2 + s^2x_1^{\,2}\}} \quad (2.45)$$

$$\simeq \frac{R_2}{s} + jX_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.46)$$

ere $R_2 \simeq \dfrac{(r_1r_2)(r_1 + r_2) + s^2r_2x_1^{\,2}}{(r_1 + r_2)^2 + s^2x_1^{\,2}}$ . . . . (2.47)

$$X_2 \simeq \frac{r_2^{\,2}x_1}{(r_1 + r_2)^2 + s^2x_1^{\,2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.48)$$

The equivalent circuit can now reduced to that shown in Fig. 2.4. Clearly Fig. 2.4 can be reduced a simple series circuit contain- an equivalent reactance, and an ivalent resistance in series.

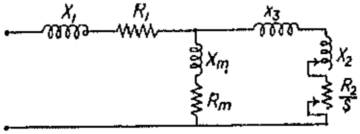


FIG. 2.4

At synchronous speed ($s = 0$)

$$\underset{(s=0)}{R_2} \simeq \frac{r_1r_2}{r_1 + r_2} = \underset{(s=0)}{R_2} \quad . \quad . \quad . \quad (2.49)$$

and $$\underset{(s=0)}{X_2} = \left(\frac{r_2}{r_1 + r_2}\right)^2 x_1 = \underset{(s=0)}{X_2} \quad . \quad . \quad . \quad (2.50)$$

At standstill, $s = 1$, and

$$\underset{(s=1)}{R_2} = \frac{r_1 r_2(r_1 + r_2) + r_2 x_1^2}{(r_1 + r_2)^2 + x_1^2} \quad . \quad . \quad . \quad (2.51)$$

$$\underset{(s=1)}{X_2} = \frac{r_2^2 x_1}{(r_1 + r_2)^2 + x_1^2} \quad . \quad . \quad . \quad (2.52)$$

$R_2(s = 1)$ is a maximum at standstill when $x_1 = r_1 + r_2$. Th is not the best condition for starting purposes, however, and does nc correspond to maximum torque at the start.

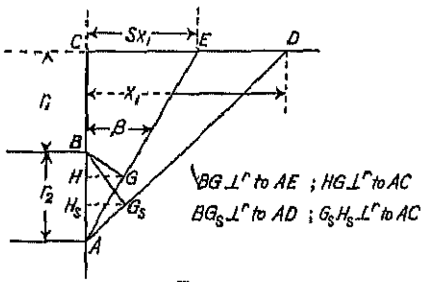


FIG. 2.5
This construction is due to Punga and Raydt

From equation (2.47) we have

$$R_2 = \frac{r_2 r_1(r_1 + r_2) + s^2 r_2 x_1^2}{(r_1 + r_2)^2 + s^2 x_1^2} \quad . \quad . \quad . \quad . \quad (2.53)$$

$$= r_2 \left[\frac{r_1(r_1 + r_2) + s^2 x_1^2}{(r_1 + r_2)^2 + s^2 x_1^2}\right] \quad . \quad . \quad . \quad (2.54)$$

$$= r_2 \left[\frac{(r_1 + r_2)^2 + s^2 x_1^2 - r_2(r_1 + r_2)}{(r_1 + r_2)^2 + s^2 x_1^2}\right] \quad . \quad (2.55)$$

$$= r_2 \left[1 - \frac{r_2(r_1 + r_2)}{(r_1 + r_2)^2 + s^2 x_1^2}\right] \quad . \quad . \quad . \quad (2.56)$$

$$= r_2 \left[1 - \frac{AB \times AC}{AE^2}\right] \quad . \quad . \quad . \quad (2.57)$$

$$= r_2 \left[1 - \frac{AB \cos \beta}{AE}\right] \quad . \quad . \quad . \quad (2.58)$$

$$= r_2 \left[1 - \frac{AG}{AE}\right] = r_2 \left[1 - \frac{AH}{AC}\right] \quad . \quad . \quad . \quad (2.59)$$

$$= r_2 \times \frac{CH}{AC} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.60)$$

In Fig. 2.5, $AB = r_2$; $BC = r_1$; CE at right angles to $AC = sx_1$; $= x_1$; BG is perpendicular to AE.

Then the equivalent resistance $R_2 = r_2 \dfrac{CH}{AC}$

As the speed varies, the point E moves from C at synchronous d to D at standstill, and the equivalent resistance of the two es $= r_2 \dfrac{CH}{AC}$; the angle $\beta =$ angle CAE.

At synchronous speed G and B coincide, and E falls on C.

At standstill
$$\underset{(s=1)}{R_2} = r_2 \frac{CH_s}{AC} \quad . \quad . \quad . \quad . \quad (2.61)$$

Now,
$$X_2 = \frac{r_2^2 x_1}{(r_1 + r_2)^2 + s^2 x_1^2} \quad . \quad . \quad . \quad (2.62)$$

$$\underset{(s=0)}{X_2} = \frac{r_2^2 x_1}{(r_1 + r_2)^2} \quad . \quad . \quad . \quad (2.63)$$

$\therefore$
$$x_1 = \underset{(s=0)}{X_2} \times \frac{(r_1 + r_2)^2}{r_2^2} \quad . \quad . \quad . \quad (2.64)$$

$\therefore$
$$X_2 = \underset{(s=0)}{X_2} \times \frac{(r_1 + r_2)^2}{(r_1 + r_2)^2 + s^2 x_1^2} \quad . \quad . \quad (2.65)$$

$\therefore$
$$X_2 = \underset{(s=0)}{X_2} \times \left(\frac{AC}{AE}\right)^2 = \underset{(s=0)}{X_2} \times \cos^2 \beta \quad . \quad . \quad (2.66)$$

If AB in Fig. 2.5 represents $X_2(s = 0)$, then
$$AH = AB \cos^2 \beta \quad . \quad . \quad . \quad (2.67)$$
represent X_2 directly.

The motion of the point H, as E moves from C to D, will give us, phically, information with regard to increase of rotor resistance decrease of rotor reactance during starting.

The currents in the two cages are dependent on the impedance ach cage, for the impedance drops are the same in each.

If I_1' and I_2' are the currents in the inner and outer cages, ectively, then
$$\frac{I_1'}{I_2'} \simeq \frac{r_2}{r_1 + jsx_1} = \frac{Z_2}{Z_1} = \frac{AB}{BE} \quad . \quad . \quad . \quad (2.68)$$
assuming negligible reactance in the outer cage.

Also
$$\frac{I_1' + I_2'}{I_1'} = \frac{AE}{AB} = \frac{I_2}{I_1'} \quad . \quad . \quad . \quad (2.69)$$

re $I_1' + I_2'$ is the vector sum of the two cage currents $= I_2$.

$\therefore$
$$I_1' : I_2' : I_2 :: AB : BE : AE$$

If the secondary current I_2 is known, the currents in the two can be read off the diagram, and also their phase relations.

The shape of the torque-slip curve will depend entirely o ratio x_1 to r_1, and also on $\frac{r_2}{r_1}$. The torque, at standstill, of the cage will be greater than at any other value of the slip, for from 0 to 1, and will be equal in synchronous watts to the l the cage.

The torque curve for the double-cage motor, and the t curves for the separate cages are shown in Fig. 2.6.

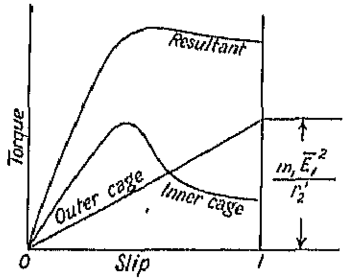


FIG. 2.6

The torque, in synchronous watts, for the inner cage

$$= \frac{sE_1^2 r_1' m_1}{(r_1')^2 + s^2(x_1')^2} \qquad . \qquad . \qquad ($$

All quantities are referred to the stator.

Neglecting x_2', the torque of the outer cage, in synchronous

$$= \frac{sE_1^2}{r_2'} \times m_1 \text{ (approx.)} \qquad . \qquad . \qquad ($$

The resultant torque, for the two cages, is

$$\frac{sE_1^2 \times r_1' m_1}{(r_1')^2 + s^2(x_1')^2} + \frac{sE_1^2 \times m_1}{r_2'} \qquad . \qquad . \ ($$

The starting torque

$$= \frac{E_1^2 \times r_1' \times m_1}{(r_1')^2 + (x_1')^2} + \frac{E_1^2 \times m_1}{r_2'} \qquad . \qquad . \ ($$

The inner cage produces but 10 to 15 per cent (approximate the starting torque. By making $\frac{x_1}{r_1}$ large, one reduces the sta current taken by the inner cage, and, since this cage produc small a part of the starting torque required, it does not matter r how one increases $\frac{x_1}{r_1}$, but it is desirable to have a reasonable valu the maximum starting torque of this cage, and this necessitate

reat a value for x_1. Clearly then one can start with the design of uter cage, by assuming it produces 80 per cent (approximately) starting torque required.

rom the relation

$$\frac{\text{starting torque}}{\text{full-load torque}} = \left(\frac{I_{scr}}{I_{2r}}\right)^2 \times \text{slip at full load}$$

$$\text{the ratio } \frac{I_{scr}}{I_{2r}} = \frac{\text{starting current in outer rotor}}{\text{full-load current in outer rotor}}$$

the ratio of the $\dfrac{\text{starting torque}}{\text{full-load torque}}$ is given, and $\dfrac{I_{scr}}{I_{2r}}$ is fixed, the f the motor at full load, considered as a single-cage motor is mined. For small ratio of $\dfrac{I_{scr}}{I_{2r}}$, clearly the slip will be relatively to produce a large starting torque. This means that r_2 must be . The slip, however, when running under full load, with the ages, will be relatively small, as one will see by inspection of the ive resistance of the two cages, given in equation (2.47).

SUMMARY OF CHAPTER II

) The squirrel-cage winding adapts itself to any number of .

) The current per ring $= \dfrac{\text{current per bar}}{2 \sin \dfrac{\pi p}{2Q_2}}$

e p = number of poles

Q_2 = total number of rotor bars

) The effective resistance per bar, including the effect of the ings, $r_b' = r_b + \dfrac{r_r}{2 \sin^2 \left(\dfrac{\pi p}{2n}\right)}$

e r_b = resistance per bar

r_r = resistance of a ring segment between two rotor bars

) Copper loss in the squirrel-cage rotor

$$= Q_2 I_b^2 \left\{ r_b + \frac{r_r}{2 \sin^2 \left(\dfrac{\pi p}{2Q_2}\right)} \right\}$$

) The effective reactance per bar, including the effect of the ance of the segment of the rings, between two bars

$$= x_b' = x_b + \frac{x_r}{2 \sin^2 \left(\dfrac{\pi p}{2Q_2}\right)}$$

where x_b = reactance of one bar at supply frequency

x_r = reactance of the segments associated with a bar at su frequency

(6) The effective *resistance* of the cage, *referred to the stator,* phase

$$R_2' = \frac{r_b' \times 4m_1 \times T_1^2 \times K_1^2 \times K_3^2}{Q_2}$$

where m_1 = number of stator phase

T_1 = turns per stator phase

K_1 = breadth factor of stator winding for the fundament

K_3 = coil span factor of stator winding for the fundamer

(7) The effective *reactance,* at supply frequency, of the

referred to the stator, $X_2' = x_b' \times \dfrac{4m_1T_1^2 \times K_1^2 \times K_3^2}{Q_2}$

(8) The squirrel-cage winding has a number of phases

$$= \frac{\text{number of rotor bars}}{\text{pairs of poles}}$$

(9) Of the rotor currents flowing under one pole, the curren half of the bars per pole flow in one direction in the end ring, the currents in the other half of the bars per pole flow in the oppo direction in the ring.

(10) To reduce noise it is desirable to skew the slots a full pitch, and to use as large an air-gap as possible.

(11) Cogging effects must be diminished, as far as possible correct choice of slot ratio. The number of slots in the rotor n differ from the number in the stator, and to reduce cogging highest common factor of the number of slots in stator and r must be as small as possible.

(12) The ordinary cage motor takes a large wattless current fi the mains at the start, and the starting torque is not large in n motors of this type.

(13) Where large starting torque, with low starting curr is required, recourse is had to the *double*-squirrel cage. The in cage has high reactance and low resistance at full frequer The outer cage has high resistance and low reactance at sup frequency.

The outer cage is very effective in producing large starting torq with low starting current. At standstill, the outer cage is the m effective. As the speed rises the reactance of the inner cage g smaller and it becomes more effective in producing its share of torque.

ig. 2.7 shows the speed/torque relationships of single-cage and ble-cage motors on the same graph.

The introduction of the double-cage is due to Dobrowolski.

The shape of the torque curve depends on the ratio of

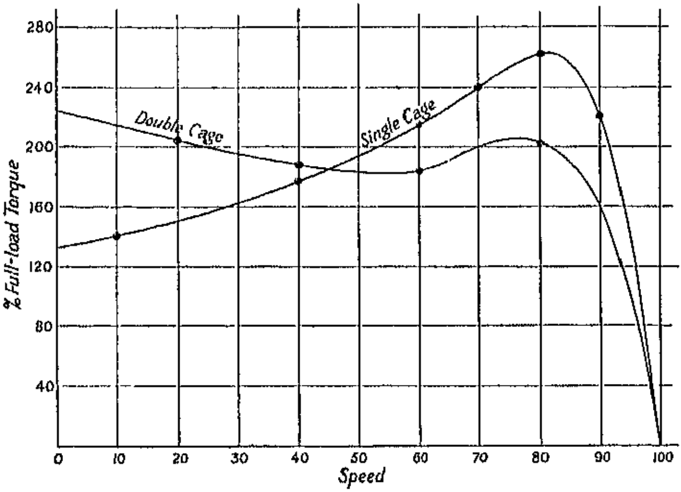


FIG. 2.7

$\frac{\text{tance of the inner cage at full supply frequency}}{\text{resistance of inner cage}}$ and also on the

of $\frac{\text{resistance of outer cage}}{\text{resistance of inner cage}}$

Usual values of $\frac{x_1}{r_1} = 4$ or 5 and $\frac{r_2}{r_1} = 2$ or 3.

Plate V (facing page 88) shows an example of a double squirrel- rotor (axially ventilated motor, with balance ring at non-fan speed 3000 r.p.m.).

CHAPTER III

The Circle Diagram

In Chapter I we developed the circle diagram by making approximations, the chief of which was the transfer of the magnetizing circuit from C and D in Fig. 1.5 to the terminals A and B. A more exact derivation will now be made.

The equivalent circuit of the polyphase motor is shown below, in Fig. 3.1. It is as follows—

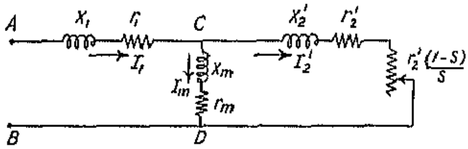


FIG. 3.1

Let the primary impedance be

$$\mathbf{Z}_1 = r_1 + jx_1 \quad . \quad . \quad . \quad . \quad (3.1)$$

The secondary impedance referred, to the primary, in the equivalent circuit, is

$$\mathbf{Z}_2' = \frac{r_2'}{s} + jx_2' \quad . \quad . \quad . \quad . \quad (3.2)$$

The magnetizing impedance

$$= \mathbf{Z}_m = r_m + jx_m \quad . \quad . \quad . \quad . \quad (3.3)$$

$$Y_m = \frac{1}{Z_m} = (g_m - jb_m) \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

$= \text{admittance of magnetizing circuit}$

$\mathbf{V}$ = applied p.d.

$\mathbf{I}_2'$ = secondary current, referred to the primary.

Then $$\mathbf{V} = \mathbf{Z}_1\mathbf{I}_1 + \mathbf{Z}_2'\mathbf{I}_2' \quad . \quad . \quad . \quad (3.5)$$

and $$\mathbf{Z}_2'\mathbf{I}_2' = +\mathbf{E}_1 = (\mathbf{I}_1 - \mathbf{I}_2')\mathbf{Z}_m \quad . \quad . \quad . \quad (3.6)$$

$\therefore$ $$\frac{\mathbf{I}_1 - \mathbf{I}_2'}{\mathbf{I}_2'} = \frac{\mathbf{Z}_2'}{\mathbf{Z}_m} \quad . \quad . \quad . \quad . \quad (3.7)$$

$$\frac{\mathbf{I}_1}{\mathbf{I}_2'} - 1 = \frac{\mathbf{Z}_2'}{\mathbf{Z}_m} \quad . \quad . \quad . \quad . \quad (3.8)$$

and $$\frac{\boldsymbol{I}_1}{\boldsymbol{I}_2'} \doteq \frac{\mathbf{Z}_2' + \mathbf{Z}_m}{\mathbf{Z}_m} \qquad (3.9)$$

and $$\boldsymbol{I}_2' = \boldsymbol{I}_1 \frac{\mathbf{Z}_m}{\mathbf{Z}_2' + \mathbf{Z}_m} \qquad (3.10)$$

Therefore, equation (3.5) becomes

$$\boldsymbol{V} = \mathbf{Z}_1 \boldsymbol{I}_1 + \frac{\mathbf{Z}_2'\mathbf{Z}_m}{\mathbf{Z}_2' + \mathbf{Z}_m} \boldsymbol{I}_1 \qquad (3.11)$$

$$\therefore \quad \boldsymbol{V} = \boldsymbol{I}_1 \left\{ \frac{\mathbf{Z}_2'\mathbf{Z}_1 + \mathbf{Z}_1\mathbf{Z}_m + \mathbf{Z}_2'\mathbf{Z}_m}{\mathbf{Z}_2' + \mathbf{Z}_m} \right\} \qquad (3.12)$$

$$\therefore \quad \boldsymbol{I}_1 = \frac{\mathbf{Z}_2' + \mathbf{Z}_m}{\mathbf{Z}_2'\mathbf{Z}_1 + \mathbf{Z}_1\mathbf{Z}_m + \mathbf{Z}_2'\mathbf{Z}_m} \times \boldsymbol{V} \qquad (3.13)$$

$$= \boldsymbol{V} \left\{ \frac{1 + \mathbf{Z}_2'\mathbf{Y}_m}{\mathbf{Z}_1 + \mathbf{Z}_2' + \mathbf{Z}_1\mathbf{Z}_2'Y_m} \right\} \qquad (3.14)$$

And from equation (3.10), we have

$$\boldsymbol{I}_2' = \boldsymbol{I}_1 \times \frac{1}{\mathbf{Z}_2'Y_m + 1}$$

$$\therefore \quad \boldsymbol{I}_2' = \boldsymbol{V} \times \frac{1}{\mathbf{Z}_1 + \mathbf{Z}_2' + \mathbf{Z}_1\mathbf{Z}_2'Ym} \qquad (3.15)$$

By substituting the values for $\mathbf{Z}_2'$, Y_m, $\mathbf{Z}_1$ we can obtain the primary and secondary currents.

Now $$Z_2'Y_m = \left(\frac{r_2'}{s} + jx_2'\right)(g_m - jb_m) \qquad (3.16)$$

$$= \frac{r_2'}{s} g_m + x_2'b_m + j\left(x_2'g_m - \frac{r_2'}{s} b_m\right) \qquad (3.17)$$

$$1 + Z_2'Y_m = 1 + \frac{r_2'}{s} g_m + x_2'b_m + j\left(x_2'g_m - \frac{r_2'}{s} b_m\right) \qquad (3.18)$$

$$= a + jb \qquad (3.19)$$

where $$a = 1 + \frac{r_2'}{s} g_m + x_2'b_m \qquad (3.20)$$

and $$b = x_2'g_m - \frac{r_2'}{s} b_m \qquad (3.21)$$

Now $\mathbf{Z}_1 + \mathbf{Z}_2' + \mathbf{Z}_1\mathbf{Z}_2'Y_m$

$$= (r_1 + jx_1) + \left(\frac{r_2'}{s} + jx_2'\right) + (r_1 + jx_1)\left(\frac{r_2'}{s} + jx_2'\right)\left(g_m - jb_m\right) \qquad (3.22)$$

$$= r_1 + \frac{r_2'}{s} + j(x_1 + x_2') + \frac{r_1 r_2'}{s} g_m + b_m x_2' r_1 + \frac{x_1 r_2'}{s} b_m - x_1 x_2' g_m$$

$$+ j\left\{x_1 \frac{r_2'}{s} g_m + x_2' r_1 g_m - \frac{r_1 r_2'}{s} b_m + x_1 x_2' b_m\right\} \quad (3.23)$$

$$= r_1 + \frac{r_2'}{s} + \frac{r_1 r_2'}{s} g_m + b_m x_2' r_1 + \frac{x_1 r_2'}{s} b_m - x_1 x_2' g_m$$

$$+ j\left\{x_1 + x_2' + \frac{x_1 r_2' g_m}{s} + x_2' r_1 g_m - \frac{r_1 r_2' b_m}{s} + x_1 x_2' b_m\right\} \quad (3.24)$$

$$= c + jd$$

$$\therefore \qquad \mathbf{I}_1 = \mathbf{V}\left\{\frac{a + jb}{c + jd}\right\} \quad . \quad . \quad . \quad . \quad (3.25)$$

$$\underset{\text{absolute}}{I_1} = V\sqrt{\frac{a^2 + b^2}{c^2 + d^2}} \quad . \quad . \quad . \quad . \quad (3.26)$$

$$I_2' = V \times \frac{1}{\sqrt{c^2 + d^2}} \quad . \quad . \quad . \quad (3.27)$$

The angle between the primary applied volts and primary current

$$= \tan^{-1}\frac{b}{a} - \tan^{-1}\frac{d}{c} \quad . \quad . \quad (3.28)$$

Now
$$a = 1 + \frac{r_2'}{s} g_m + x_2' b_m \quad . \quad . \quad . \quad (3.29)$$

$$b = x_2' g_m - \frac{r_2' b_m}{s} \quad . \quad . \quad . \quad (3.30)$$

$$c = r_1 + \frac{r_2'}{s} + \frac{r_1 r_2'}{s} g_m + b_m x_2' r_1 + \frac{x_1 r_2'}{s} b_m - x_1 x_2' g_m \quad (3.31)$$

$$d = x_1 + x_2' + \frac{x_1 r_2' g_m}{s} + x_2' r_1 g_m - \frac{r_1 r_2' b_m}{s} + x_1 x_2' b_m \quad (3.32)$$

$$\frac{a + jb}{c + jd} = \frac{(r_2' g_m - j r_2' b_m) + 1\,(1 + x_2' b_m + j x_2' g_m)s}{r_2' + r_1 r_2' g_m + x_1 r_2' b_m + j(x_1 r_2' g_m - r_1 r_2' b_m) + \{r_1 + b_m x_2' r_1 - x_1 x_2' g_m + j(x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m)\}s} \quad . \quad . \quad (3.33)$$

$$= \frac{\mathbf{E} + \mathbf{F}s}{\mathbf{G} + \mathbf{H}s}$$

where
$$\mathbf{E} = r_2' g_m - j r_2' b_m \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.34)$$

$$\mathbf{F} = 1 + x_2' b_m + j x_2' g_m \quad . \quad . \quad . \quad . \quad . \quad (3.35)$$

$$\mathbf{G} = r_2' + r_1 r_2' g_m + x_1 r_2' b_m + j(x_1 r_2' g_m - r_1 r_2' b_m) \quad . \quad (3.36)$$

$$\mathbf{H} = r_1 + b_m x_2' r_1 - x_1 x_2' g_m + j(x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m) \quad (3.37)$$

Now

$$\frac{\mathbf{E} + \mathbf{F}s}{\mathbf{G} + \mathbf{H}s} = \frac{\mathbf{F}}{\mathbf{H}} + \frac{\mathbf{E} - \dfrac{\mathbf{F}\mathbf{G}}{\mathbf{H}}}{\mathbf{G} + \mathbf{H}s} \quad . \quad . \quad . \quad (3.38)$$

$$= \mathbf{A} + \frac{\mathbf{B}}{\mathbf{G} + \mathbf{H}s} \quad . \quad . \quad . \quad (3.39)$$

Now $\mathbf{A} = \dfrac{\mathbf{F}}{\mathbf{H}}$ = a complex constant

$$= \frac{1 + x_2{}'b_m + jx_2{}'g_m}{r_1 + b_m x_2{}' r_1 - x_1 x_2{}' g_m + j(x_1 + x_2{}' + x_2{}' r_1 g_m + x_1 x_2{}' b_m)} \quad (3.40)$$

$$\text{i.e. } \mathbf{A} = \frac{\{1 + x_2{}'b_m + jx_2{}'g_m\} \times \{r_1 + b_m x_2{}' r_1 - x_1 x_2{}' g_m - j(x_1 + x_2{}' + x_2{}' r_1 g_m + x_1 x_2{}' b_m)\}}{(r_1 + b_m x_2{}' r_1 - x_1 x_2{}' g_m)^2 + (x_1 + x_2{}' + x_2{}' r_1 g_m + x_1 x_2{}' b_m)^2} \quad . \quad . \quad (3.41)$$

$$\mathbf{A} = f + jg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.42)$$

where

$$f = \frac{(r_1 + b_m x_2{}' r_1 - x_1 x_2{}' g_m)(1 + x_2{}' b_m) + (x_1 + x_2{}' + x_2{}' r_1 g_m + x_1 x_2{}' b_m) x_2{}' g_m}{(r_1 + b_m x_2{}' r_1 - x_1 x_2{}' g_m)^2 + (x_1 + x_2{}' + x_2{}' r_1 g_m + x_1 x_2{}' b_m)^2} \quad (3.43)$$

$$g = \frac{x_2{}' g_m \{r_1 + b_m x_2{}' r_1 - x_1 x_2{}' g_m\} - \{1 + x_2{}' b_m\} \times \{x_1 + x_2{}' + x_2{}' r_1 g_m + x_1 x_2{}' b_m\}}{(r_1 + b_m x_2{}' r_1 - x_1 x_2{}' g_m)^2 + (x_1 + x_2{}' + x_2{}' r_1 g_m + x_1 x_2{}' b_m)^2} \quad (3.44)$$

$\mathbf{B} = \mathbf{E} - \dfrac{\mathbf{F}\mathbf{G}}{\mathbf{H}}$ also a complex constant.

$$\mathbf{B} = r_2{}' g_m - j r_2{}' b_m - \frac{\{1 + x_2{}' b_m + jx_2{}' g_m\}\{r_2{}' + r_1 r_2{}' g_m + x_1 r_2{}' b_m + j(x_1 r_2{}' g_m - r_1 r_2{}' b_m)\}}{r_1 + b_m x_2{}' r_1 - x_1 x_2{}' g_m + j(x_1 + x_2{}' + x_2{}' r_1 g_m + x_1 x_2{}' b_m)} \quad . \quad (3.45)$$

$$= h + jk \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.46)$$

where h is the real part of $\mathbf{B}$ and k is the imaginary part of $\mathbf{B}$.

$$\left.\begin{aligned} \text{Let } \mathbf{E} &= q + jr \\ \mathbf{G} &= y + jz \\ \mathbf{F} &= s_1 + jt \\ \mathbf{H} &= u + jv \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.47)$$

then
$$\mathbf{B} = q + jr - \frac{(s_1 + jt)(y + jz)}{u + jv} \qquad (3.48)$$

$$= \frac{uq + jur + jqv - vr - s_1 y + tz - jty - js_1 z}{u + jv} \qquad (3.49)$$

$$= \frac{uq - vr - s_1 y + tz + j(ur - ty - s_1 z + qv)}{u + jv} \qquad (3.50)$$

$$= \frac{(uq - vr - s_1 y + tz) + j(ur + qv - ty - s_1 z)}{u + jv}$$

Rationalizing the denominator, we have

$$\mathbf{B} = \frac{\{(uq - vr - s_1 y + tz) + j(ur + qv - ty - s_1 z)\}\{u - jv\}}{u^2 + v^2} \qquad (3.51)$$

$$= \frac{u^2 q - uvr - us_1 y + utz + vur + v^2 q - vty - vs_1 z}{u^2 + v^2}$$
$$+ \frac{j(u^2 r + uqv - uty - us_1 z - uvq + v^2 r + vs_1 y - vtz)}{u^2 + v^2} \qquad (3.52)$$

$$= h + jk$$

$$h = \frac{u^2 q - uvr - us_1 y + utz + vur + v^2 q - vty - vs_1 z}{u^2 + v^2} \qquad (3.53)$$

$$k = \frac{u^2 r + uqv - uty - us_1 z - uvq + v^2 r + vs_1 y - vtz}{u^2 + v^2} \qquad (3.54)$$

Also
$$\mathbf{A} = \frac{\mathbf{F}}{\mathbf{H}} = \frac{s_1 + jt}{u + jv} = \frac{(s_1 + jt)(u - jv)}{u^2 + v^2} \qquad (3.55)$$

$$= \frac{us_1 + tv + j(tu - s_1 v)}{u^2 + v^2} \qquad (3.56)$$

$$= \frac{us_1 + tv}{u^2 + v^2} + j\,\frac{(tu - s_1 v)}{u^2 + v^2} \qquad (3.57)$$

$$= f + jg \qquad (3.58)$$

where
$$f = \frac{us_1 + tv}{u^2 + v^2} \qquad (3.59)$$

$$g = \frac{tu - s_1 v}{u^2 + v^2} \qquad (3.60)$$

Now
$$\mathbf{I}_1 = \mathbf{V}\left[\frac{\mathbf{E} + \mathbf{F}s}{\mathbf{G} + \mathbf{H}s}\right] = \mathbf{V}\left[\mathbf{A} + \frac{\mathbf{B}}{\mathbf{G} + \mathbf{H}s}\right] \qquad (3.61)$$

$$= \mathbf{V}\,(f + jg) + \mathbf{V}\,\frac{(h + jk)}{\mathbf{G} + \mathbf{H}s} \qquad (3.62)$$

Let us take a system of rectangular co-ordinates, in which the vertical axis represents real quantities, which we will call the X-axis, and the horizontal axis represents imaginary quantities, i.e. the quantities which are multiplied by j or $-j$.

Then $\boldsymbol{G} + \boldsymbol{H}s$ represents a straight line (Fig. 3.2).

$$\begin{aligned}\boldsymbol{G} + \boldsymbol{H}s &= r_2' + r_1r_2'g_m + x_1r_2'b_m + j(x_1r_2'g_m - r_1r_2'b_m) \\ &\quad + \{r_1 + b_mx_2'r_1 - x_1x_2'g_m + j(x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m)\}s \\ &= y + jz + (u + jv)s \,. \qquad \qquad (3.63)\end{aligned}$$

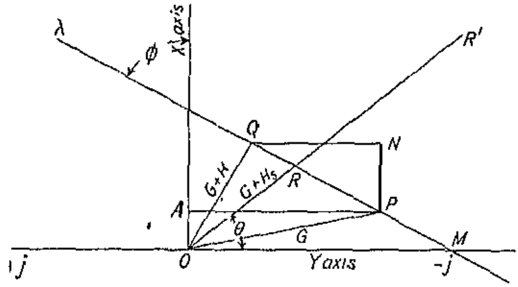


FIG. 3.2

Set off $OA = y = r_2' + r_1r_2'g_m + x_1r_2'b_m$ vertically along the X-axis, and $z = AP = x_1r_2'g_m - r_1r_2'b_m$ parallel to the $-j$ axis, for AP is negative.

Then $OP = \boldsymbol{G}$.

Now draw PN vertically parallel to the X-axis and make it $= u$, and NQ vertically parallel to the Y-axis $= v$.

Then $PQ = \boldsymbol{H}$ for $s = 1$. Therefore, $OQ = \boldsymbol{G} + \boldsymbol{H}$ and represents $\boldsymbol{G} + \boldsymbol{H}s$ for $s = 1$.

As s varies from 0 to 1, R moves from P to Q. Take any point R between P and Q, and join OR and produce to R', where $OR' = \dfrac{1}{OR}$. Then R' is said to be the inverse point of R.

Repeat this construction, i.e. find the various inverse points corresponding to the various positions of R on the line PQ and PQ produced. The loci of the various inverse points is a circle, which passes through the origin O.

We notice also that since $OR = \boldsymbol{G} + \boldsymbol{H}s$ is a complex quantity, we may represent it by $re^{j\theta}$, then if r' represents OR' in magnitude

$$\frac{1}{\boldsymbol{G} + \boldsymbol{H}s} = \frac{1}{r}\,e^{-j\theta} = r'e^{-j\theta} \qquad (3.64)$$

Thus, if the vector OR makes an angle θ with the horizontal, the point R' (i.e. the inverse point) should be equal to $\dfrac{1}{r}$ and be below the horizontal by the angle $-\theta$.

Thus, the complete inversion is a circle, passing through the origin O, and is an image of the circle above the horizontal axis in the horizontal axis. In other words our inverse circle is below the Y-axis. Let y and x represent the co-ordinates of point R and y' and x' represent the co-ordinates of point R'.

$$OR = r,\ OR' = r'.$$

Then $$x' = r' \sin \theta = \frac{1}{r} \sin \theta = \frac{r \sin \theta}{r^2} = \frac{x}{r^2} = \frac{x}{x^2 + y^2} \quad . \quad (3.65)$$

$$y' = r' \cos \theta = \frac{1}{r} \cos \theta = \frac{r \cos \theta}{r^2} = \frac{y}{r^2} = \frac{y}{x^2 + y^2} \quad . \quad (3.66)$$

Thus, for the inverse of any locus $F(x, y) = 0$, we have the locus $F\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) = 0$.

Therefore, to obtain the inverse, we substitute in the equation of the curve $\frac{x}{x^2 + y^2}$ for x and $\frac{y}{x^2 + y^2}$ for y.

Now we saw that **G** + ***Hs*** represents the line PQT. Let the equation of this line be represented by

$$y = mx + n \quad . \quad . \quad . \quad . \quad (3.67)$$

The inverse of this line is

$$\frac{y}{x^2 + y^2} = \frac{mx}{x^2 + y^2} + n \quad . \quad . \quad . \quad (3.68)$$

i.e. $$- mx - n(x^2 + y^2) + y = 0 \quad (3.69)$$

i.e. $$x^2 + y^2 + \frac{mx}{n} - \frac{y}{n} = 0 \quad . \quad . \quad . \quad (3.70)$$

i.e. $$\left(x + \frac{m}{2n}\right)^2 + \left(y - \frac{1}{2n}\right)^2 = \frac{m^2}{4n^2} + \frac{1}{4n^2} \quad . \quad . \quad (3.71)$$

$$= \frac{m^2 + 1}{4n^2} \quad . \quad . \quad (3.72)$$

This is the equation of a circle. The radius

$$= \frac{\sqrt{m^2 + 1}}{2n} \quad . \quad . \quad . \quad . \quad (3.73)$$

The x_0 co-ordinate of the centre, with respect to O,

$$= \frac{-m}{2n} \quad . \quad . \quad . \quad . \quad (3.74)$$

and $$y_0 = \frac{1}{2n} \quad . \quad . \quad . \quad . \quad (3.75)$$

Now we need the radius in amps, and the other co-ordinates in amps.

Now $$\boldsymbol{I}_1 = \boldsymbol{V}\{f + jg\} + \boldsymbol{V}\frac{(h + jk)}{\boldsymbol{G} + \boldsymbol{H}s} \qquad (3.76)$$

The radius of the circle in amps

$$= V\sqrt{h^2 + k^2}\,\frac{\sqrt{m^2 + 1}}{2n} \qquad (3.77)$$

The true origin will be displaced from O.

The real co-ordinate of the centre, i.e. the x-co-ordinate, will be

$$Vf + V\sqrt{h^2 + k^2}\left(\frac{-m}{2n}\right) \qquad (3.78)$$

i.e. x_c is multiplied by $V\sqrt{h^2 + k^2}$

The new y-co-ordinate of the centre

$$= Vg + V\sqrt{h^2 + k^2}\,\frac{1}{2n} \qquad (3.79)$$

i.e. y_0 is multiplied by $\sqrt{h^2 + k^2}$ and the radius is turned through an angle $\alpha = \tan^{-1}\dfrac{k}{h}$

To determine the radius, we need to know the values of m and n.

Now m, the slope of the line $PQT = \tan\phi$.

$$m = \frac{x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m}{r_1 + b_m x_2' r_1 - x_1 x_2' g_m} \qquad (3.80)$$

And n is the intercept OM on the $-j$ axis.

We have $$y - y_1 = m(x - x_1)$$

where x_1 and y_1 are the co-ordinates of P in Fig. 3.2,

i.e. $$y = mx - mx_1 + y_1 \qquad (3.81)$$

When $x = 0$, $y = OM$,

$$\therefore \quad OM = y_1 - mx_1 = n$$

$$\therefore\ n = x_1 r_2' g_m - r_1 r_2' b_m - \frac{(x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m)(r_2' + r_1 r_2' g_m + x_1 r_2' b_m)}{r_1 + b_m x_2' r_1 - x_1 x_2' g_m}$$

We have seen that

$$\boldsymbol{I}_1 = \boldsymbol{V}\left[\frac{\boldsymbol{E} + \boldsymbol{F}s}{\boldsymbol{G} + \boldsymbol{H}s}\right] \qquad (3.82)$$

At synchronous speed, $s = 0$

$$\text{and } \underset{(s=0)}{I_1} = V \times \frac{E}{G} = V \frac{(r_2' g_m - j r_2' b_m)}{r_2' + r_1 r_2' g_m + x_1 r_2' b_m + j(x_1 r_2' g_m - r_1 r_2' b_m)}$$

$$\therefore \underset{(s=0)}{I_1} = \bar{V} \sqrt{\frac{(r_2' g_m)^2 + (r_2' b_m)^2}{(r_2' + r_1' r_2' g_m + x_1 r_2' b_m)^2 + (x_1 r_2' g_m - r_1 r_2' b_m)^2}} \quad (3.83)$$

The angle between the primary current and primary volts at synchronous speed

$$= -\tan^{-1} \frac{r_2' b_m}{r_2' g_m} - \tan^{-1} \frac{x_1 r_2' g_m - r_1 r_2' b_m}{r_2' + r_1 r_2' g_m + x_1 r_2' b_m} \quad . \quad (3.84)$$

At standstill, $s = 1$,

$$\underset{(s=1)}{I_1} = V \left[\frac{E + F}{G + H} \right]$$

$$\therefore \underset{(s=1)}{I_1} = V \left[\frac{r_2' g_m - j r_2' b_m + 1 + x_2' b_m + j x_2' g_m}{\begin{array}{l} r_2' + r_1 r_2' g_m + x_1 r_2' b_m + r_1 + b_m x_2' r_1 - x_1 x_2' g_m \\ + j(x_1 r_2' g_m - r_1 r_2' b_m + x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m) \end{array}} \right]$$

$$\therefore \underset{(s=1)}{I_1} = \bar{V} \sqrt{\frac{(r_2' g_m + 1 + x_2' b_m)^2 + (x_2' g_m - r_2' b_m)^2}{\begin{array}{l} (r_2' + r_1 r_2' g_m + x_1 r_2' b_m + r_1 + b_m x_2' r_1 - x_1 x_2' g_m)^2 \\ + (x_1 r_2' g_m - r_1 r_2' b_m + x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m)^2 \end{array}}} \quad . \quad . \quad (3.85)$$

The angle between the primary current and primary volts

$$= \tan^{-1} \frac{x_2' g_m - r_2' b_m}{1 + r_2' g_m + x_2' b_m} - \tan^{-1} \frac{x_1 r_2' g_m - r_1 r_2' b_m + x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m}{r_2' + r_1 r_2' g_m + x_1 r_2' b_m + r_1 + b_m x_2' r_1 - x_1 x_2' g_m} \quad (3.86)$$

At $s = \infty$

$$\underset{(s=\infty)}{I_1} = V \frac{F}{H} = \frac{V \times \{1 + x_2' b_m + j x_2' g_m\}}{r_1 + b_m x_2' r_1 - x_1 x_2' g_m + j(x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m)} \quad . \quad . \quad (3.87)$$

and

$$\underset{(s=\infty)}{I_1} = \bar{V} \sqrt{\frac{(1 + x_2' b_m)^2 + (x_2' g_m)^2}{(r_1 + b_m x_2' r_1 - x_1 x_2' g_m)^2 + (x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m)^2}} \quad . \quad . \quad (3.88)$$

and the angle between primary volts and primary current for $s = \infty$

$$= \tan^{-1} \frac{x_2' g_m}{1 + x_2' b_m} - \tan^{-1} \frac{x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m}{r_1 + b_m x_2' - x_1 x_2' g_m} \quad (3.89)$$

Thus, three points on the circle can be calculated and the centre and the radius found by geometry or by calculation.

Some of these expressions are rather cumbrous, but in any given example, we shall be able to discard, without sensible error, those which are negligibly small.

Thus, $r_2'b_m \simeq \dfrac{r_2'}{x_m}$ is small, and $x_1r_2g_m$ very small, and other similar terms.

The expressions can then be considerably simplified by neglecting these small terms.

The Torque Line

We will now show that the torque is represented by the vertical intercept between the circle and a straight line, called the torque line.

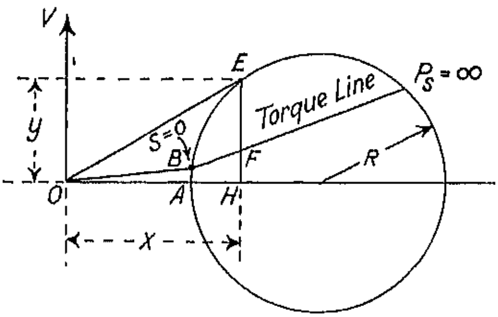


Fig. 3.3

Let B be the point at which the slip $s = 0$, and, therefore, the speed is synchronous. At B no torque is developed in the rotor. The primary current at B is OB.

Let P be the point where $s = \infty$. Since the torque in synchronous watts per phase $= (I_2')^2 \dfrac{r_2'}{s}$, the torque is zero when $s = \infty$.

Let x and y be the co-ordinates of point E. Then

$$(x - x_c)^2 + (y - y_c)^2 = R^2$$

where x_c and y_c are the co-ordinates of the centre and R = radius of the circle.

$$\therefore \qquad x^2 + y^2 = R^2 + 2xx_c + 2yy_c - x_c^2 - y_c^2$$

The torque in synchronous watts = input to the rotor and the input to the rotor = input to the stator — iron loss — stator copper loss

$$= V \times EH - (x^2 + y^2)r_1 - C \qquad . \qquad . \quad (3.90)$$

where C = iron loss

and OE = primary amps $= \sqrt{x^2 + y^2}$

$$\therefore \qquad Vy - (x^2 + y^2)r_1 - C = \text{rotor input per phase}$$

$$\therefore \qquad Vy - r_1[R^2 + 2xx_c + 2yy_c - x_c^2 - y_c^2) - C = 0 \quad . \quad (3.91)$$

i.e. $$y(V - 2y_0r_1) - 2xx_0r_1 - r_1[R^2 - x_0^2 - y_0^2) - C = 0 \quad . \quad (3.92)$$

At B and P, the torque is zero, since at B, $I_2' = 0$ and at P, $s = \infty$, so at B also the torque is zero. But this condition implies the truth of equation (3.92). This equation is linear in y and x, and, therefore, represents a straight line.

We may also arrive at the circle and its centre co-ordinates in another way—

We have $$\boldsymbol{I}_1 = \boldsymbol{V}\left\{\frac{a + jb}{c + jd}\right\} = \boldsymbol{V}\left\{\frac{\mathbf{E} + \mathbf{F}s}{\mathbf{G} + \mathbf{H}s}\right\} \quad . \quad . \quad (3.33)$$

Let $\boldsymbol{I}_1 = \alpha - j\beta$

also
$$\begin{aligned}
\mathbf{E} &= r_2'g_m - jr_2'b_m = q + jr \\
\mathbf{F} &= 1 + x_2'b_m + jx_2'g_m = s_1 + jt \\
\mathbf{G} &= r_2' + r_1r_2'g_m + x_1r_2'b_m + j(x_1r_2'g_m - r_1r_2'b_m) = y + jz \\
\mathbf{H} &= r_1 + b_mx_2'r_1 - x_1x_2'g_m + j(x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m) \\
&= u + jv
\end{aligned}$$

From equation (3.33) we have

$$\boldsymbol{I}_1\mathbf{G} + \boldsymbol{I}_1\mathbf{H}s = V\mathbf{E} + V\mathbf{F}s \quad . \quad . \quad . \quad (3.93)$$

i.e. $$\boldsymbol{I}_1\mathbf{G} - V\mathbf{E} = s(V\mathbf{F} - \boldsymbol{I}_1\mathbf{H}) \quad . \quad . \quad . \quad (3.94)$$

$$\therefore\ (\alpha - j\beta)(y + jz) - V(q + jr) = s[Vs_1 + Vjt - (\alpha - j\beta)(u + jv)] \quad . \quad . \quad (3.95)$$

$$\therefore\ \alpha y + \beta z + j(z\alpha - y\beta) - Vq - Vjr$$
$$= s(Vs_1 - \alpha u - \beta v) + js[Vt + \beta u - \alpha v] \quad (3.96)$$

$$\therefore \qquad \alpha y + \beta z - Vq = s(Vs_1 - \alpha u - \beta v) \quad . \quad . \quad (3.97)$$

$$\alpha z - y\beta - Vr = s(Vt + \beta u - \alpha v) \quad . \quad . \quad (3.98)$$

$$\therefore \qquad \frac{\alpha y + \beta z - Vq}{\alpha z - y\beta - Vr} = \frac{Vs_1 - \alpha u - \beta v}{Vt + \beta u - \alpha v} \quad . \quad . \quad (3.99)$$

$$\alpha yVt + \alpha y\beta u - \alpha^2yv + \beta zVt + \beta^2zu - \beta z\alpha v - V^2qt - V\beta qu + \alpha vVq$$
$$= \alpha zVs_1 - \alpha^2zu - \alpha\beta zv - y\beta Vs_1 + y\beta\alpha u + y\beta^2v - V^2rs_1 + V\alpha ru + V\beta rv$$

i.e. $$\alpha^2[zu - yv] + \beta^2[zu - yv] + \beta[zVt - Vqu + yVs_1 - Vrv]$$
$$+ \alpha[yVt + vVq - zVs_1 - Vru] + V^2rs_1 - V^2qt = 0 \quad . \quad (3.100)$$

i.e. that is

$$\alpha^2 + \beta^2 + \alpha\left[\frac{yVt + vVq - zVs_1 - Vru}{zu - yv}\right]$$
$$+ \beta\left[\frac{zVt - Vqu + yVs_1 - Vrv}{zu - yv}\right] + V^2\left[\frac{rs_1 - qt}{zu - yv}\right] = 0 \quad (3.101)$$

Let $K = \dfrac{yVt + vVq - zVs_1 - Vru}{zu - yv}$ . . . . . (3.102)

$$l = \frac{zVt - Vqu + yVs_1 - Vrv}{zu - yv} \quad (3.102)$$

$$N = V^2 \left(\frac{rs_1 - qt}{zu - yv}\right) \quad (3.104)$$

Our equation is, therefore,

$$\alpha^2 + \beta^2 + K\alpha + l\beta + N = 0 \quad (3.105)$$

i.e.

$$\left(\alpha + \frac{1}{2}K\right)^2 + \left(\beta + \frac{l}{2}\right)^2 = \frac{K^2}{4} + \frac{l^2}{4} - N \quad (3.106)$$

The co-ordinates of the centre are

$$-\frac{1}{2}K \text{ and } -\frac{l}{2}$$

and the radius of the circle

$$= \sqrt{\frac{K^2}{4} + \frac{l^2}{4} - N} \quad (3.107)$$

CHAPTER IV

The Single-Phase Induction Motor

THIS motor is in very wide use in industry, especially in the fractional horse-power field, and is extensively used for oil-burners, fans, blowers, office appliances, and certain types of small tools. It is also widely used for refrigerators, pumps, compressors, washing and ironing machines. For use on the farm, it is used in sizes of from 1 to 7·5 h.p., for threshing, feed-grinding, corn-shelling, grain-drying, corn-husking, hay-baling, milking, water-pumping, bottle-washing, milk-cooling, and also for silo work. It is a very useful motor in relatively small outputs. For large powers it suffers from disadvantages, which are inherent in its characteristics, and is never used in cases where a polyphase motor can be adopted. Chief among these disadvantages are: (*a*) output only about 50 per cent of the three-phase motor, for a given frame size and temperature rise, (*b*) lower power factor, (*c*) lower efficiency, and (*c*) has no inherent starting torque, and, therefore, requires a starting winding, with a phase-splitting device. In spite of these drawbacks, it is admirably adapted for small outputs. Whilst the simplest in construction of all a.c. machines, its theory is more complicated than any. Its characteristics are deduced by two very different methods. The methods are known as: (1) the cross-field theory and (2) the two revolving fields theory.

THE CROSS-FIELD THEORY*

This theory is due to Potier and H. Goerges. Since this offers some difficulty to the average student, it will help, perhaps, if a few simple fundamental principles are enunciated. Briefly these are—

(*a*) In any transformer, the secondary ampere-turns oppose the primary ampere-turns, and tend to reduce the flux.

(*b*) To determine the direction of the e.m.f. in any conductor, place the *right* hand along the conductor, in such a position that the flux passes from the palm to the back and the thumb points in the direction of motion of the conductor relative to the field, then the direction of *the e.m.f. is from the wrist to the finger tips.*

(*c*) To determine the direction of the force on a conductor, carrying current, and situated in a magnetic field, place the *left hand*

* The following paragraphs follow Behrend closely

along the conductor, in such a position that the flux passes from the palm to the back, and the current flows in the direction from the wrist to the finger tips, then the thumb points to the direction of the force and motion, due to the force.

(*d*) Another variant of (*c*) is to imagine the flux to be like a heavy curtain, the field surrounding the current-carrying conductors has a general direction at right angles to the main flux, and tends to move the curtain to the right or left, depending on its direction. The force acting on the conductor is in the opposite direction to the movement of the curtain, due to the field of the conductor. This is shown in Fig. 4.1.

The single-phase motor has usually, a distributed, drum winding, which usually fills two-thirds of the pole pitch in each pole arc.

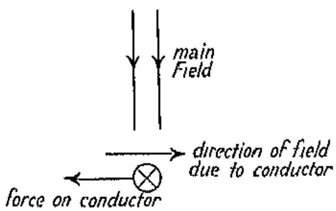


FIG. 4.1

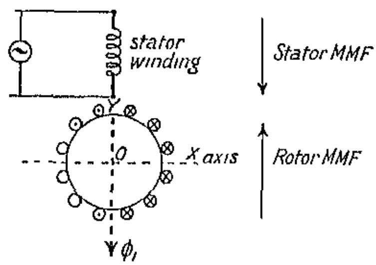


FIG. 4.2

The remaining one-third of the pole arc, accommodates another winding, which is used for starting purposes.

In Fig. 4.2 we represent the stator winding by a single coil, whose magnetic axis is the vertical axis. With the direction of current, at the instant chosen, our magnetic flux is shown. We will call this flux ϕ_1. Assume there is a squirrel-cage rotor.

At standstill our rotor behaves like a short-circuited transformer, so our rotor currents, at standstill, must flow in such a direction that the rotor m.m.f. opposes that of the stator. The magnetic axis of the rotor, due to these currents, generated in it by transformer action, is the vertical or *Y*-axis or *transformer axis.*

It will be noted that, *at standstill,* there is no resultant torque on the rotor, for the interaction of the currents on the right-hand side of the rotor with the flux ϕ_1 is neutralized by the action of the currents on the left-hand side of the rotor with ϕ_1.

Thus, the single-phase induction motor, unaided by a starting winding, has no starting torque.

Such a motor, however, if given motion in any direction, either clockwise or anti-clockwise, will develop a torque and continue to run in that direction. Now let us assume it is running in a clockwise direction. Rotation in the flux ϕ_1 will generate in the rotor conductors e.m.f.s, and currents will flow in such a direction as to produce

another flux along the horizontal axis, i.e. the X-axis. This is known as the *excitation axis*.

Fig. 4.3 shows the directions of e.m.f.s and currents in the rotor due to rotation in the transformer flux during the time ϕ_1 acts downwards. We will concentrate our attention on the actions in the two axes X and Y.

Consider first the X-axis. We have e.m.f.s set up by rotation in ϕ_1, in the direction shown in Fig. 4.3. Due to pulsation of ϕ_2, we have a transformer e.m.f. in the X-axis, which lags 90° behind ϕ_2.

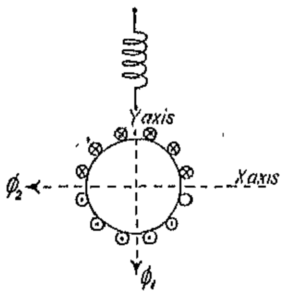


FIG. 4.3

We shall call the e.m.f.s in the X-axis—

(*a*) Rotational e.m.f., e_{xr}.

(*b*) Transformer e.m.f., e_{xt}.

These two e.m.f.s in the X-axis oppose one another, and are nearly in phase opposition. The direction of the currents set up is determined by the rotational voltage e_{xr}.

In the vertical axis, the Y-axis, we have also two e.m.f.s—

(*a*) A transformer e.m.f., e_{yt}, due to pulsation of ϕ_1.

(*b*) A rotational e.m.f., e_{yr}, due to rotation in the flux ϕ_2.

$e_{yt} > e_{yr}$ and the direction of the currents in the Y-axis is determined by the resultant of e_{yt} and e_{yr}.

Now in the Y-axis e_{yt} and e_{yr} are opposed to one another, or more strictly, the phase angle between them is greater than 90°.

Now torque will be produced in the following manner—

(*a*) By the interaction of the excitation flux ϕ_2 and the currents in the Y-axis. This is the *driving torque*, produced by the interaction of *the currents in the Y-axis* with *the excitation flux* ϕ_2.

(*b*) By the interaction of the currents in the excitation axis, i.e. the X-axis currents, with the transformer flux ϕ_1. This is a *retarding torque.*

Thus, we note that in the single-phase motor there are two torques produced—

(1) The driving torque, due to the action of the currents i_y and ϕ_2, the excitation flux.

(2) The retarding torque, due to the action of the currents i_x and ϕ_1, the transformer flux.

Now the excitation flux ϕ_2 is displaced by $\frac{\pi}{2}$ in space from ϕ_1, but its time phase angle will differ from $\frac{\pi}{2}$.

The current produced in the X-axis by rotation in ϕ_1 will lag behind the rotational e.m.f. in the X-axis by an angle whose tangent $= \dfrac{\text{rotor reactance}}{\text{resistance}}$, and since the rotational voltage is in phase with

the flux ϕ_1, it follows that the time phase angle of ϕ_1 and ϕ_2 will be given by $\tan^{-1}\frac{x_2}{r_2} = \tan^{-1}\frac{\text{rotor reactance}}{\text{rotor resistance}}$.

We shall be concerned with the relation of ϕ_1 and ϕ_2 very closely in what follows. It should be noted that ϕ_2 is in phase, neglecting hysteresis effects, with i_x.

At synchronous speed $\phi_1 = \phi_2$, and if the time phase angle were $\frac{\pi}{2}$, we should have a true rotating wave of flux of constant amplitude, as in the polyphase motor.

Since the time phase angle $= \tan^{-1}\frac{x_2}{r_2}$, it follows that the angle will approach $\frac{\pi}{2}$ the smaller r_2, i.e. the smaller the *rotor resistance.* Since, also, x_2 depends on the leakage flux of the rotor, it follows that, since the leakage flux is inversely proportional to the reluctance, this

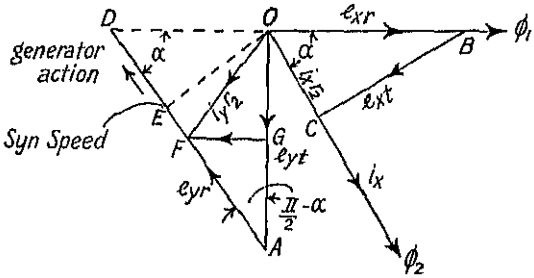


FIG. 4.4

reluctance must be small also. *A low resistance of rotor* is required for approximate quadrature time relations of the fluxes ϕ_1 and ϕ_2. Whatever the resistance be, the phase relation between ϕ_1 and ϕ_2 remains approximately constant as the load and speed vary.

The vector diagram, Fig. 4.4, is very simple. We shall adopt the device of replacing leakage reactance voltage drops by the corresponding leakage flux. In that case, only resistance drops are present.

$OB = e_{xr}$ = rotational e.m.f. in X-axis in phase with ϕ_1

$\alpha = \angle BOC = \tan^{-1}\frac{x_2}{r_2}$ = constant angle between ϕ_1 and ϕ_2

$OC = i_x r_2$ = resistance voltage in x-axis of the rotor

$BC = e_{xt}$ = transformer voltage in X-axis lagging $\frac{\pi}{2}$ behind ϕ_2

$OA = e_{yt}$ = transformer voltage in Y-axis due to ϕ_1 and lagging $\frac{\pi}{2}$ behind ϕ_1

$AF = e_{yr}$ = rotational voltage in Y-axis due to rotation in ϕ_2

$OF = i_y r_2$ = resistance voltage in the vertical axis

OBC = triangle of voltages in horizontal or excitation axis (= X-axis)

OAF = triangle of voltages in vertical Y-axis of rotor

As the load varies, the point F moves along the line AD. At the point E, where OE is perpendicular to AD, synchronous speed is reached.

From E to D and beyond we have generator action.

Relation of Fluxes ϕ_1 and ϕ_2

Let f = supply frequency = pairs of poles × sychronous revolutions per second

f_1 = frequency due to rotation = pairs of poles × revolutions per second

Then
$$e_{xr} \propto \phi_1 \times f_1 \qquad (4.1)$$

and
$$e_{xt} \propto \phi_2 \times f = e_{xr} \sin \alpha \qquad (4.2)$$

$$\therefore \quad \phi_2 \times f = \phi_1 \times f_1 \sin \alpha \qquad (4.3)$$

$$\therefore \quad \frac{\phi_2}{\phi_1} = \frac{f_1}{f} \sin \alpha = k \sin \alpha \qquad (4.4)$$

where $k = \dfrac{f_1}{f}$

Also
$$e_{yt} \propto \phi_1 \times f \qquad (4.5)$$

$$e_{yr} \propto \phi_2 \times f_1 \propto \phi_1 \times \left(\frac{f_1}{f}\right)^2 \times f \sin \alpha \qquad (4.6)$$

$$AE = OA \cos\left(\frac{\pi}{2} - \alpha\right) = OA \sin \alpha = e_{yt} \sin \alpha \qquad (4.7)$$

i.e.
$$AE = e_{yt} \sin \alpha \propto \phi_1 f \sin \alpha \qquad (4.8)$$

$$AF = e_{yr} \propto \phi_1 \times \left(\frac{f_1}{f}\right)^2 f \sin \alpha \qquad (4.9)$$

$AF = AE$ when $f_1 = f$, i.e. at synchronous speed. Therefore, the point E corresponds to synchronous speed. Also it is clear that, beyond the point E, the current i_y makes an angle less than 90° with e_{yr}, i.e. the machine is a generator beyond E towards D.

The driving torque, multiplied by the speed, expressed in watts

$$= e_{yr} \times i_y \cos \angle OFA \propto AF \times EF \qquad (4.10)$$

At synchronous speed the $\triangle OCB = \triangle OAE$.

Now
$$OE = OA \cos \alpha = e_{yt} \cos \alpha. \qquad (4.11)$$

and $$OC = OB \cos \alpha = e_{xr} \cos \alpha \,. \qquad (4.12)$$

but $e_{yt} = e_{xr}$ at synchronous speed (*see* equations (4.1) and (4.5)).

Also $$OE^2 = AE \,.\, DE \qquad (4.13)$$

The negative power

$$= e_{xr} i_x \cos \alpha \qquad (4.14)$$

$$= i_x^2 r_2 \propto \frac{OC^2}{r_2} \qquad (4.15)$$

Now $$OC = e_{xr} \cos \alpha \propto \phi_1 \times f_1 \cos \alpha$$

$$OE = OA \cos \alpha \propto \phi_1 f \cos \alpha \qquad (4.16)$$

$$\therefore \quad OC = OE \times \frac{f_1}{f} \qquad (4.17)$$

Now $$AF = e_{yr} \propto \phi_2 \times f_1 \propto \phi_1 \times \left(\frac{f_1}{f}\right)^2 f \sin \alpha \,. \qquad (4.18)$$

$$AE = e_{yt} \sin \alpha \propto \phi_1 \times f \sin \alpha \qquad (4.19)$$

$$\therefore \quad \frac{AF}{AE} = \left(\frac{f_1}{f}\right)^2 \qquad (4.20)$$

Now the copper loss $\propto OC^2 \propto OE^2 \times \left(\frac{f_1}{f}\right)^2$

i.e. $$OC^2 \propto AE \,.\, ED \times \frac{AF}{AE} \qquad (4.21)$$

for $OE^2 = AE \,.\, ED$

$$\therefore \quad OC^2 \propto AF \,.\, ED$$

Therefore, the negative power is proportional to

$$AF \times ED \text{ in Fig. 4.4} \qquad (4.22)$$

Therefore, the net power = driving power − negative power

$$\propto AF(EF - ED) \text{ in Fig. 4.4} \qquad (4.23)$$

Equivalent Circuit

The current in the Y-axis, $i_y = \frac{OF}{r_2}$, is resolved into two components, $\frac{OG}{r_2}$ is in phase with OA, i.e. with e_{yt}. It is, therefore, a watt component, and a second component $\frac{FG}{r_2}$, which is wattless, and subtracting in its magnetizing action from the flux ϕ_1. (*See* Fig. 4.4.)

Now $\quad OG = OA - AG = e_{vt} - AG = e_{vt} - AF \sin\alpha \quad . \quad (4.24)$

$$= e_{vt} - e_{vr} \sin\alpha \quad . \quad (4.25)$$

Let R = resistance, which when acted upon by e_{vt} will have a current OG flowing in it, i.e. $\frac{e_{vt}}{R}$ = the watt component of the current.

$$GF = AF\cos\alpha = e_{vr}\cos\alpha \quad . \quad . \quad . \quad (4.26)$$

but
$$e_{vr} = \left(\frac{f_1}{f}\right)^2 \sin\alpha\, e_{vt}$$

$\therefore$
$$GF = e_{vt}\left(\frac{f_1}{f}\right)^2 \sin\alpha\cos\alpha$$

and
$$OG = e_{vt} - \left(\frac{f_1}{f}\right)^2 \sin^2\alpha\, e_{vt} = e_{vt}\left\{1 - \left(\frac{f_1}{f}\right)^2 \sin^2\alpha\right\} . \quad (4.27)$$

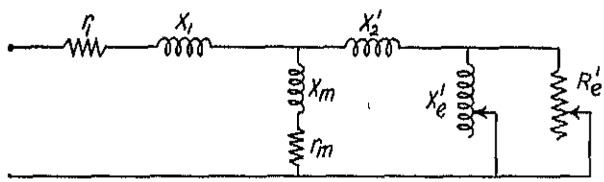


Fig. 4.5

$\therefore$
$$\frac{e_{vt}}{R} = \frac{e_{vt}\left\{1 - \left(\frac{f_1}{f}\right)^2 \sin^2\alpha\right\}}{r_2} \quad . \quad . \quad . \quad (4.28)$$

$\therefore$
$$R_e = \frac{r_2}{1 - \left(\frac{f_1}{f}\right)^2 \sin^2\alpha} \quad . \quad . \quad . \quad (4.29)$$

Therefore, the equivalent resistance

$$= \frac{r_2}{1 - \left(\frac{f_1}{f}\right)^2 \sin^2\alpha} \quad . \quad . \quad (4.30)$$

The wattless component of current in the Y-axis

$$= \frac{FG}{r_2} = e_{vt}\left(\frac{f_1}{f}\right)^2 \frac{\sin\alpha\cos\alpha}{r_2} \quad . \quad . \quad (4.31)$$

Therefore, let X_e = the equivalent reactance.

Then
$$\frac{e_{vt}}{X_e} = e_{vt}\left(\frac{f_1}{f}\right)^2 \frac{\sin\alpha\cos\alpha}{r_2} \quad . \quad . \quad . \quad (4.32)$$

$$\therefore \qquad X_o = \frac{r_2}{\left(\frac{f_1}{f}\right)^2 \sin\alpha\cos\alpha} \qquad (4\cdot33)$$

The equivalent reactance

$$= \frac{r_2}{\left(\frac{f_1}{f}\right)^2 \sin\alpha\cos\alpha} \qquad (4\cdot34)$$

$$= \frac{r_2}{k^2 \sin\alpha\cos\alpha}$$

The total current in the Y-axis, $i_y = \frac{OF}{r_2}$

$$= \frac{\sqrt{OG^2 + GF^2}}{r_2} \qquad (4\cdot35)$$

$$OG^2 = e_{yt}^2 [1 - 2k^2\sin^2\alpha + k^4\sin^4\alpha] \qquad (4\cdot36)$$

where $$k = \frac{f_1}{f} = \frac{\text{actual speed}}{\text{synchronous speed}} \qquad (4\cdot37)$$

$$GF^2 = e_{yt}^2 k^4 \sin^2\alpha\cos^2\alpha$$

$$\therefore i_y = \frac{e_{yt}\sqrt{1 - 2k^2\sin^2\alpha + k^4\sin^4\alpha + k^4\sin^2\alpha\cos^2\alpha}}{r_2} \qquad (4\cdot38)$$

$$= \frac{e_{yt}\sqrt{1 - 2k^2\sin^2\alpha + k^4\sin^2\alpha(\sin^2\alpha + \cos^2\alpha)}}{r_2} \qquad (4\cdot39)$$

$$= \frac{e_{yt}\sqrt{1 - 2k^2\sin^2\alpha + k^4\sin^2\alpha}}{r_2} \qquad (4\cdot40)$$

$$= \frac{e_{yt}\sqrt{1 + k^2\sin^2\alpha(k^2 - 2)}}{r_2} \qquad (4\cdot41)$$

When F is at A, i.e. $k = \frac{f_1}{f} = 0$, i.e. at standstill

$$\underset{(k=0)}{i_y} = \frac{e_{yt}}{r_2} \qquad (4\cdot42)$$

At E, when F coincides with E, i.e. at synchronous speed

$$\underset{(k=1)}{i_y} = \frac{e_{yt}\cos\alpha}{r_2} \qquad (4\cdot43)$$

This is the minimum secondary current as will be seen clearly from Fig. 4.4, since OE is the shortest distance from O to AD.

The equivalent reactance of the rotor circuit X_c (*see* equation (4·34))

$$= \frac{r_2}{\left(\frac{f_1}{f}\right)^2 \sin\alpha\cos\alpha} = \frac{r_2}{k^2 \sin\alpha\cos\alpha} \quad . \quad . \quad (4{\cdot}44)$$

The equivalent resistance

$$= \frac{r_2}{1 - k^2 \sin^2\alpha} \quad . \quad . \quad . \quad . \quad (4{\cdot}45)$$

Since the locus of the rotor current in the Y-axis is a circle, it follows that the locus of the primary current is a circle, for the primary current will consist of the magnetizing current OM (in Fig. 4.6), which

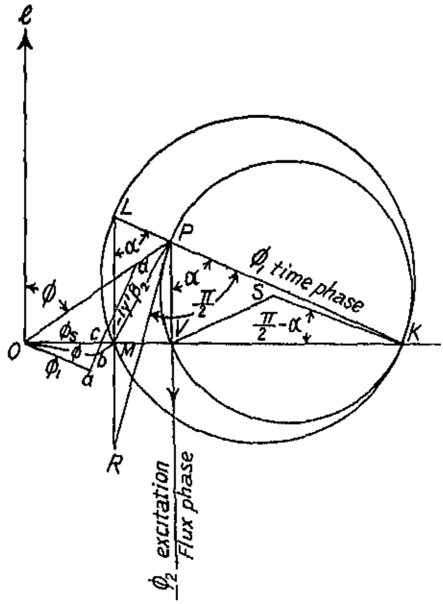


FIG. 4.6

is necessary to produce the flux ϕ_s and a component, compounded with OM, which is parallel to I_y, the current in the Y-rotor circuit, and drawn in the opposite direction to I_y, and the magnetizing action of this component, flowing in the stator must equal the magnetizing action of I_y. This component is shown as MP in Fig. 4.6.

$$MP = -I_y \times \frac{\text{rotor turns per pole (effective)}}{\text{stator turns per pole (effective)}} \times \beta_2 \quad (4{\cdot}46)$$

At synchronous speed, the transformer flux ϕ_1 and the excitation flux ϕ_2 are equal and $OM = MV$.

The ideal short-circuit current $= OK$.

As the load varies, the locus of the primary current is the circle KPV.

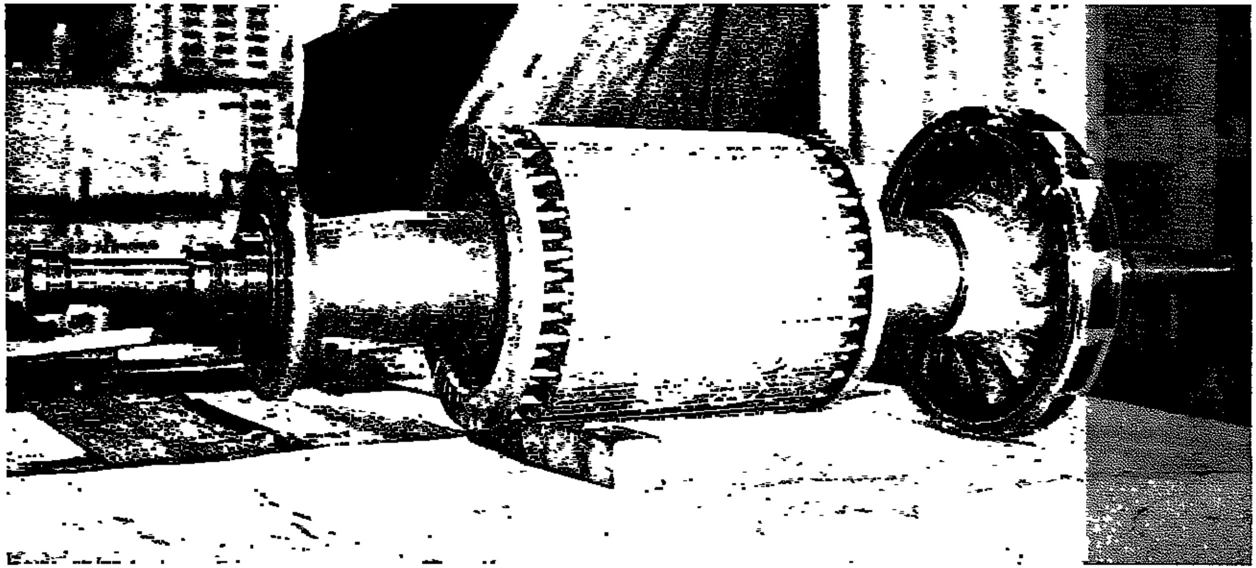
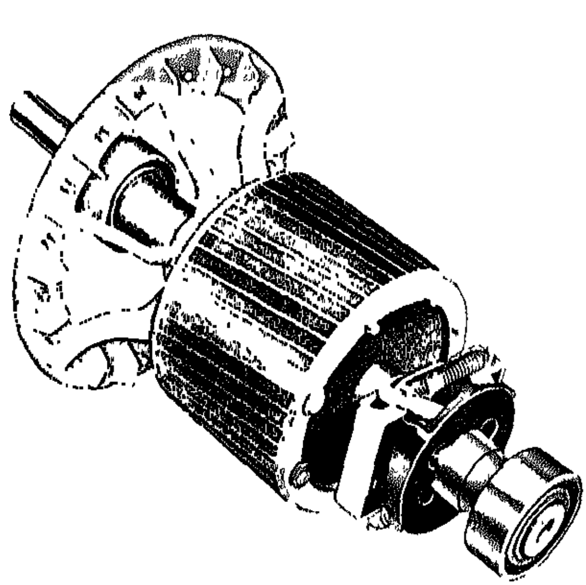


PLATE X. DOUBLE SQUIRREL-CAGE ROTOR

This is for an axially ventilated motor with a balance ring at the non-fan end. Speed 3000 r.p.m.

(Courtesy, English Electric Co., Ltd.)



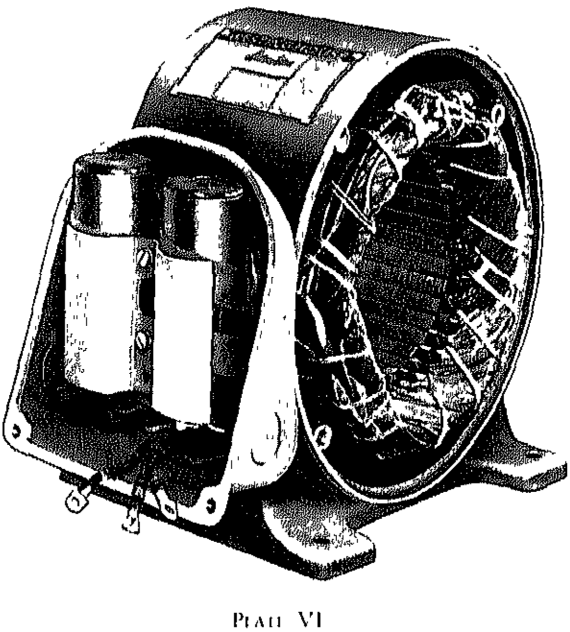


Plate VI

(*Upper*) Squirrel-cage Rotor

Capacitor conveniently located inside the protecting conduit box

(*Courtesy Westinghouse Electric Corp. of America*)

(*Lower*) Single-phase Capacitor Motor Stator

This had a cast aluminum fan, centrifugal starting switch and sealed, prelubricated bearings. Note balance washers on end ring

(*Courtesy Westinghouse Electric Corp. of America*)

OP = primary current, and represents to another scale the fictitious flux linking the primary due to current OP. Of this total flux linking the primary, the part Od crosses the gap and links the secondary.

ad = rotor current I_y referred to the primary, and represents to another scale the flux produced by the secondary current. Of this flux the part ab is secondary leakage flux, and the part bd or MP combines with OP to produce OM the resultant flux linking the primary winding, OM.

The resultant of Od, i.e. that part of the primary flux which crosses the air-gap, and da the rotor flux due to rotor current MP, gives us the flux linking the secondary, namely $Oa = \phi_1$—our transformer flux.

The resultant of ϕ_1 and ab, the secondary leakage flux, gives Ob, the resultant flux crossing the gap and linking both primary and secondary. bM parallel to OP and equal to dP represents the primary leakage flux. The vector sum of ϕ and bM gives ϕ_s, the flux linking the primary, to which the voltage $-e$, lagging 90° behind ϕ_s, is due.

Now PK is parallel to Oa or ϕ_1, the transformer flux, and MP is parallel to ac. The triangles Oac and MPK are similar. PV is parallel to ϕ_2, the excitation flux, and $\angle KPV = \angle KLM$.

The $\triangle MPR$ is the $\triangle AOF$ in Fig. 4.4.

$PR = OA = e_{yt}$; $PM = i_y r_2$; $MR = FA = e_{yr}$ drawn vertically parallel to ϕ_2.

The $\angle VSK = 2\alpha$, and since $SK = SV$ = radius of the circle, it follows that $\angle SKV = \angle SVK = \frac{\pi}{2} - \alpha$.

So the centre of the circle is found by drawing lines from K and V, making angles of $\frac{\pi}{2} - \alpha$ with the horizontal.

Let $ad = \beta_2 bd$

and $\quad OP = \beta_1 Od$

Then ab (Fig. 4.6) $= (\beta_2 - 1)bd$

and
$$bc = bd - cd$$
$$= bd\left(1 - \frac{1}{\beta_1}\right)$$
$$ab + bc = \left(\beta_2 - \frac{1}{\beta_1}\right) bd = \left(\frac{\beta_1\beta_2 - 1}{\beta_1}\right) bd = ac \quad . \quad (4.47)$$
$$\therefore \quad \frac{ac}{cd} = \left(\frac{\beta_1\beta_2 - 1}{\beta_1}\right)\frac{bd \times \beta_1}{bd} = \beta_1\beta_2 - 1 \quad . \quad . \quad (4.48)$$

but from similar triangles Oac and MPK

$$\frac{Oc}{ca} = \frac{MK}{MP}$$

$$\therefore \qquad \frac{Oc}{MK} = \frac{ca}{MP}$$

$$\therefore \qquad \frac{OM}{\beta_1 MK} = \frac{ca}{\beta_1 cd}$$

i.e.

$$\frac{OM}{MK} = \frac{ca}{cd} = \beta_1\beta_2 - 1$$

If we call this ratio σ, we have

$$\frac{OM}{MK} = \sigma \qquad (4.49)$$

Since

$$\frac{MV}{VK} = \frac{LP}{PK} = \frac{\phi_1}{\beta_2}\bigg/ MK\beta_1 \frac{\phi_1}{\phi_s} \qquad (4.50)$$

$$= \frac{1}{\beta_1\beta_2}\left(\frac{\phi_s}{MK}\right) \qquad (4.51)$$

$$= \frac{1}{\beta_1\beta_2}(\beta_1\beta_2 - 1) \qquad (4.52)$$

$$= \frac{\sigma}{\sigma + 1} \qquad (4.53)$$

and

$$\frac{MK}{VK} = \frac{2\sigma + 1}{\sigma + 1} = \frac{MV + VK}{VK} = 1 + \frac{MV}{VK} \qquad (4.54)$$

$$= 1 + \frac{\sigma}{\sigma + 1} \qquad (4.55)$$

$$\frac{OM}{MV} = \frac{OM}{MK} \times \frac{MK}{MV} = \sigma \times \frac{MK}{VK} \times \frac{VK}{MV} \qquad (4.56)$$

$$= \sigma \times \frac{2\sigma + 1}{\sigma + 1} \times \frac{\sigma + 1}{\sigma}$$

$$= 2\sigma + 1 \qquad (4.57)$$

Also $$\frac{OV}{OM} = \frac{MV}{OM} + 1 = \frac{1}{2\sigma + 1} + 1 = \frac{2\sigma + 2}{2\sigma + 1} = \frac{2\{\sigma + 1\}}{2\sigma + 1} \qquad (4.58)$$

Note that

$$\frac{MV}{OM} = \frac{1}{2\sigma + 1} \qquad (4.59)$$

i.e. the magnetizing current for the excitation field demands an extra wattless current in the Y-circuit of the stator = the magnetizing current for $\phi_s \times \dfrac{1}{1 + 2\sigma}$

Since σ is usually small the *magnetizing current* for the single-phase induction motor is nearly double that necessary to produce ϕ_1, the transformer field.

Note, we have used for σ here the ratio $\frac{OM}{MK}$. When considering the polyphase motor we used for σ the ratio

$$\frac{OM}{OK} = \frac{\text{magnetizing current}}{\text{ideal short-circuit current}} = \frac{i_\mu}{i_{sc}}$$

We shall call the ratio $\frac{OM}{OK} = \sigma_1$.

Then $$\frac{OM}{MK} = \frac{OM}{OK - OM} = \frac{\frac{OM}{OK}}{1 - \frac{OM}{OK}} = \frac{\sigma_1}{1 - \sigma_1} \quad . \quad (4.60)$$

$\therefore$ $$\sigma = \frac{\sigma_1}{1 - \sigma_1} \quad . \quad . \quad . \quad . \quad (4.61)$$

Thus, if we desire to keep to our old definition, the relation is

$$\sigma = \frac{\sigma_1}{1 - \sigma_1}$$

where $\sigma = \frac{OM}{MK}$

and $\sigma_1 = \frac{OM}{OK}$

Finally, we have

$$\begin{aligned}\frac{OV}{VK} &= \frac{OM + MV}{VK} = \frac{OM}{VK} + \frac{MV}{VK} \\ &= \frac{OM}{MV} \times \frac{MV}{VK} + \frac{MV}{VK} \\ &= 2\sigma + 1 \times \frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} \\ &= \frac{\sigma}{\sigma + 1}\{2\sigma + 1 + 1\} = 2\sigma \quad . \quad (4.62)\end{aligned}$$

Thus, collecting our results—

$$\frac{OM}{MK} = \sigma = \beta_1\beta_2 - 1 \quad . \quad . \quad . \quad (4.63)$$

where β_1 = stator leakage factor
and β_2 = rotor leakage factor

$$\frac{OM}{MK} = \sigma = \frac{\sigma_1}{1 - \sigma_1} \quad . \quad . \quad . \quad (4.64)$$

where $\sigma_1 = \dfrac{OM}{OK}$

$$\frac{MV}{VK} = \frac{\sigma}{\sigma + 1} = \sigma_1 \qquad (4.65)$$

$$\frac{OM}{MV} = 2\sigma + 1 = \frac{\sigma_1 + 1}{1 - \sigma_1} \qquad (4.66)$$

$$\frac{OV}{OM} = 2\left\{\frac{\sigma + 1}{2\sigma + 1}\right\} = \frac{2}{1 + \sigma_1} \qquad (4.67)$$

$$\frac{MV}{OM} = \frac{1}{2\sigma + 1} = \frac{1 - \sigma_1}{1 + \sigma_1} \qquad (4.68)$$

$$\frac{OV}{VK} = 2\sigma = \frac{2\sigma_1}{1 - \sigma_1} \qquad (4.69)$$

To make clear these various equations, it should be noted that

$$\left.\begin{aligned} \beta_1 &= \frac{OP}{od} = \frac{\text{total flux produced by stator current}}{\text{that part of the stator flux which crosses the gap and links the secondary}} \\ \text{and } \beta_2 &= \frac{ad}{bd} = \frac{\text{total flux produced by rotor current}}{\text{flux produced by rotor current which crosses the gap and links the primary}} \end{aligned}\right\} \qquad (4.70)$$

It will be abundantly clear now why the power factor of this machine is lower than the three- or two-phase motor

We now have an additional magnetizing current MV, which causes a raising of the dispersion coefficient σ_1, and as we have seen this is one of the main causes of low power factor.

FURTHER ANALYSIS*

Consider a coil AB on the rotor, the plane of which makes an angle θ with the X-axis at any time t.

Let us consider two axes, viz. the X-axis or transformer axis and the Y-axis or excitation axis.

Let ϕ_c = resolved part of the flux in the direction of the coil axis.

Then $$\phi_c = \phi_{RX} \sin\theta - \phi_{RY} \cos\theta \qquad (4.71)$$

where ϕ_{RX} = flux in the rotor along the X-axis . . . (4.72)

ϕ_{RY} = flux in the rotor along the Y-axis . . . (4.73)

* This analysis is from a paper by the author in the *Journal of the American Institute of Electrical Engineers*, vol. 67 (1948)

The e.m.f. induced in the coil AB, due to variation of ϕ_{RX} and ϕ_{RY} with time, and also due to variation of θ with time is obtained by differentiating equation (4.71) with respect to time

$$\therefore \phi_c' = \omega \cos \theta \phi_{RX} + \omega \sin \theta \phi_{RY} + \sin \theta \phi_{RX}' - \cos \theta \phi_{RY}' \quad (4{\cdot}74)$$

where $\omega = \dfrac{d\theta}{dt}$

The e.m.f. in the coil $= -\phi_c'$ and the current in the coil

$$= \frac{-\phi_c'}{r \times 10^8} \text{ amps} = i_c \quad . \quad . \quad . \quad (4{\cdot}75)$$

In this analysis, the flux linking the coil, is the resultant of the flux crossing the air-gap into the rotor and the leakage flux. In other words the leakage reactance drops are taken care of, and the

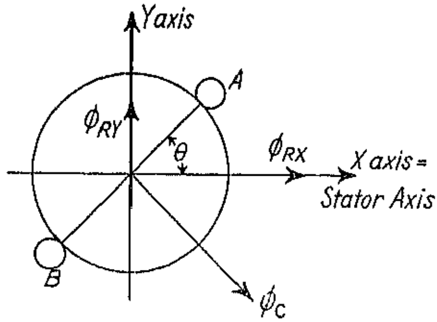


Fig. 4.7

only voltage drop to consider is that used in overcoming resistance.
r = resistance of one coil, in ohms.

The m.m.f. of the coil

$$= 0{\cdot}4\pi i_c = F_c \quad . \quad . \quad . \quad . \quad (4{\cdot}76)$$

$$\therefore F_c = \left(\frac{-4\pi}{10^9 r}\right)(\omega\phi_{RX} \cos \theta - \phi_{RY}' \cos \theta + \omega\phi_{RY} \sin \theta + \phi_{RX}' \sin \theta) \quad . \quad . \quad (4{\cdot}77)$$

The resolved part of F_c along the X-axis, which we may call

$$\delta F_{cRX} = \left(\frac{-4\pi}{10^9 r}\right)(\omega\phi_{RX} \sin \theta \cos \theta - \phi_{RY}' \sin \theta \cos \theta + \omega\phi_{RY} \sin^2 \theta + \phi_{RX}' \sin^2 \theta) \quad . \quad (4{\cdot}78)$$

To find the total m.m.f. along the X-axis, due to N_2 coils or $2N_2$ rotor bars, we must sum up the m.m.f.s due to these N_2 coils along the X-axis.

$$\therefore \quad F_{RX} = \left(\frac{-4\pi}{10^9 r}\right) N_2 \int_0^{\pi} [\sin^2 \theta\{\omega\phi_{RY} + \phi_{RX}'\} + \sin \theta \cos \theta(\omega\phi_{RX} - \phi_{RY}')] \frac{d\theta}{\pi} \quad . \quad (4{\cdot}79)$$

$$= \frac{-2\pi N_2\{\omega\phi_{RY} + \phi_{RX}'\}}{10^9 r} \quad . \quad . \quad . \quad . \quad . \quad (4.80)$$

This is the m.m.f. along the X-axis due to all the coils on the rotor.

Similarly, resolving F_0 along the Y-axis and summing up as before, over the coils of the rotor, we get

$$F_{RY} = \frac{2\pi N_2}{10^9 r}\{\omega\phi_{RX} - \phi_{RY}'\} \quad . \quad . \quad (4.81)$$

F_{RY} is the m.m.f. along the Y-axis, due to all the rotor coils.

Let $k_2 = \frac{2\pi N_2}{10^9 r}$, then equations (4.80) and (4.81) may be written

$$F_{RX} = -k_2\omega\phi_{RY} - k_2\phi_{RX}' \quad . \quad . \quad (4.82)$$

and

$$F_{RY} = +k_2\omega\phi_{RX} - k_2\phi_{RY}' \quad . \quad . \quad (4.83)$$

Let us call the flux produced in the X-axis of the stator ϕ_{SX}: this ϕ_{SX} is the transformer flux produced by the combined m.m.f.s of stator and rotor in the X-axis, which is the magnetic axis of the stator.

Let ϕ_{SY} = stator flux in the Y-axis, i.e. the excitation axis

and ϕ_{RY} = rotor flux in the Y-axis

Then

$$\phi_{SY} = \frac{\omega\hat{\phi}_{SX}\sin(\omega_0 t - \lambda)}{\sqrt{\left[\left\{(1+\sigma_r)k_2\left(\frac{\sigma_r + \sigma_r\sigma_s + \sigma_s}{R}\right)\right\}\{\omega^2 - \omega_0^2\} + \frac{R(1+\sigma_s)}{k_2}\right]^2 + [1 + 2\sigma_r + 2\sigma_s + 2\sigma_r\sigma_s]^2\omega_0^2}} \quad . \quad . \quad (4.84)$$

$$\text{and } \tan\lambda = \frac{(1 + 2\sigma_r + 2\sigma_s + 2\sigma_r\sigma_s)\omega_0}{\left\{(1+\sigma_r)k_2\left(\frac{\sigma_r + \sigma_r\sigma_s + \sigma_s}{R}\right)\right\}\left\{(\omega^2 - \omega_0^2) + \frac{R(1+\sigma_s)}{k_2}\right\}} \quad . \quad . \quad (4.85)$$

where λ = angle of lag of ϕ_{SY} behind ϕ_{SX}

ω = angular velocity of rotor in radians per second

$\omega_0 = 2\pi \times$ supply frequency

$\sigma_r = \dfrac{\text{leakage flux in rotor}}{\text{useful flux in rotor}}$

$\sigma_s = \dfrac{\text{leakage flux in the stator}}{\text{useful flux in the stator}}$

R = air-gap reluctance

Proof of the Equation for ϕ_{SY} in Terms of ϕ_{SX}

$$F_{RX} = -k_2\omega\phi_{RY} - k_2\phi_{RX}' \quad . \quad . \quad . \quad (4.82)$$

and

$$F_{RY} = k_2\omega\phi_{RX} - k_2\phi_{RY}' \quad . \quad . \quad . \quad (4.83)$$

ϕ_{LX} = stator plus rotor leakage flux in the X-axis

$$= F_{SX}P_{SX} - F_{RX}P_{r\alpha} \quad . \quad . \quad . \quad (4.86)$$

P_{SX} = equivalent permeance of stator leakage path.

$P_{r\alpha}$ = equivalent permeance of rotor leakage path.

Now F_{RX} measured along the X-axis will be negative, or opposed to F_{SX}.

$$\phi_{RY} = \phi_{SY} + F_{RY}P_{RY} = \phi_{SY} + F_{RY}P_{r\alpha} \quad . \quad (4.87)$$

$$= \frac{F_{RY}}{R} + F_{RY}P_{r\alpha} \quad . \quad . \quad . \quad . \quad . \quad (4.88)$$

$$= F_{RY}\frac{(1 + \sigma_r)}{R} \quad . \quad . \quad . \quad . \quad . \quad (4.89)$$

$$\therefore \quad \frac{\phi_{RY}R}{1 + \sigma_r} = F_{RY}$$

$$= k_2\omega[\phi_{SX} - F_{SX}P_{SX} + F_{RX}P_{rx}] - k_2\phi_{RY}' \quad (4.90)$$

$$\phi_{SX} = \frac{(F_{SX} + F_{RX})}{R}(1 + \sigma_s) \quad . \quad . \quad (4.91)$$

$$\frac{R\phi_{SX}}{(1 + \sigma_s)} - F_{RX} = F_{SX} \quad . \quad . \quad . \quad (4.92)$$

$$\therefore \quad \frac{\phi_{RY}R}{1 + \sigma_r} = k_2\omega\left[\phi_{SX} - \frac{R\phi_{SX}P_{SX}}{1 + \sigma_s} + F_{RX}P_{SX} + F_{RX}P_{rx}\right] - k_2\phi_{RY}' \quad . \quad . \quad (4.93)$$

$$= k_2\omega\left[\phi_{SX} - \frac{\sigma_s\phi_{SX}}{1 + \sigma_s} + F_{rx}\left(\frac{\sigma_s}{R} + \frac{\sigma_r}{R}\right)\right] - k_2\phi_{RY} \quad . \quad (4.94)$$

$$= \frac{k_2\omega\phi_{SX}}{1 + \sigma_s} + \left\{k_2\omega\frac{\sigma_s}{R} + k_2\omega\frac{\sigma_r}{R}\right\}F_{RX} - k_2\phi_{RY}' \quad . \quad (4.95)$$

Note

$$P_{SX}R = \sigma_s \text{ and } P_{r\alpha}R = \sigma_r \quad . \quad . \quad (4.96)$$

$$\therefore \quad \frac{\phi_{RY}R}{1 + \sigma_r} = \frac{k_2\omega\phi_{SX}}{1 + \sigma_s} + \left\{k_2\omega\frac{\sigma_s}{R} + k_2\omega\frac{\sigma_r}{R}\right\}$$

$$\{-k_2\omega\phi_{RY} - k_2\phi_{RX}'\} - k_2\phi_{RY}' \quad . \quad (4.97)$$

$$= A\phi_{SX} - B\phi_{RY} - C\phi_{RX}' - k_2\phi_{RY}' \quad . \quad . \quad (4.98)$$

where
$$A = \frac{k_2\omega}{1 + \sigma_s} \qquad (4.99)$$

$$B = k_2\omega\left[\frac{[k_2\omega\sigma_s}{R} + \frac{k_2\omega\sigma_r}{R}\right] \qquad (4.100)$$

$$C = \frac{B}{\omega} \qquad (4.101)$$

Now substitute the value of ϕ_{RX} (derived from equation (4.83)),

viz.
$$\phi_{RX} = \frac{F_{RY}}{k_2\omega} + \frac{\phi_{RY}'}{\omega} \qquad \text{(from equation (4.83))}$$

$$= \frac{\phi_{RY}R}{(1 + \sigma_r)k_2\omega} + \frac{\phi_{RY}'}{\omega}$$

$$\therefore \qquad \phi_{RX}' = \frac{\phi_{RY}'R}{(1 + \sigma_r)(k_2\omega)} + \frac{\phi_{RY}''}{\omega} \qquad (4.102)$$

Therefore, equation (4.100) becomes

$$A\phi_{SX} = \frac{C\phi_{RY}''}{\omega} + \phi_{RY}'\left\{\frac{CR}{k_2\omega(1 + \sigma_r)} + k_2\right\} + \phi_{RY}\left\{\frac{R}{1 + \sigma_r} + B\right\} \qquad (4.103)$$

$$\therefore\ A\phi_{SX} = \left[\frac{CD^2}{\omega} + \left\{\frac{CR}{k_2\omega(1 + \sigma_r)} + k_2\right\}D + \left(\frac{R}{1 + \sigma_r} + B\right)\right]\phi_{RY} \qquad (4.104)$$

$$\therefore\ \phi_{RY} = \frac{A\omega\phi_{SX}}{CD^2 + \left(\dfrac{CR}{k_2(1 + \sigma_r)} + k_2\omega\right)D + \dfrac{R\omega}{1 + \sigma_r} + B\omega} \qquad (4.105)$$

here $\phi' = \frac{d}{dt}\phi = D\phi$.

$$D^2\phi_{RY} = \frac{d^2}{dt^2}(\phi_{RY})$$

ϕ_{SX} is assumed to vary sinusoidally,

i.e.
$$\phi_{SX} = \hat{\phi}_{SX}\sin\omega_0 t$$

$$\phi_{RY} = \frac{\omega\hat{\phi}_{SX}\sin\omega_0 t}{\dfrac{C}{A}D^2 + \left(\dfrac{CR}{Ak_2(1 + \sigma_r)} + \dfrac{k_2\omega}{A}\right)D + \dfrac{1}{A}\left\{\dfrac{R\omega}{1 + \sigma_r} + B\omega\right\}} \qquad (4.106)$$

Now
$$\frac{C}{A} = \frac{B}{\omega A} = \frac{k_2\omega\left[\dfrac{k_2\omega\sigma_s}{R} + \dfrac{k_2\omega\sigma_r}{R}\right](1 + \sigma_s)}{k_2\omega \times \omega} \qquad (4.107)$$

$$\simeq \frac{k_2\{\sigma_r + \sigma_r\sigma_s + \sigma_s\}}{R} \qquad (4.108)$$

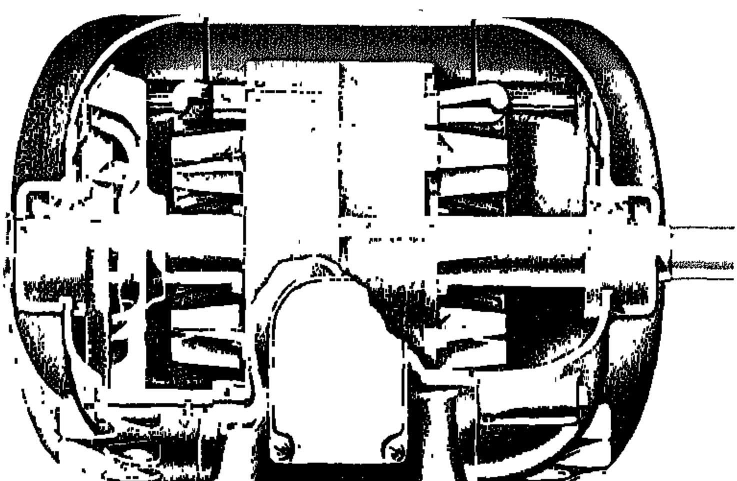


Plate VII

(*Upper*) Single-phase Motor Type CAP

(*Courtesy Westinghouse Electric Corp. of America*)

(*Lower*) Splash-proof Type Life-line Motor

(*Courtesy Westinghouse Electric Corp. of America*)

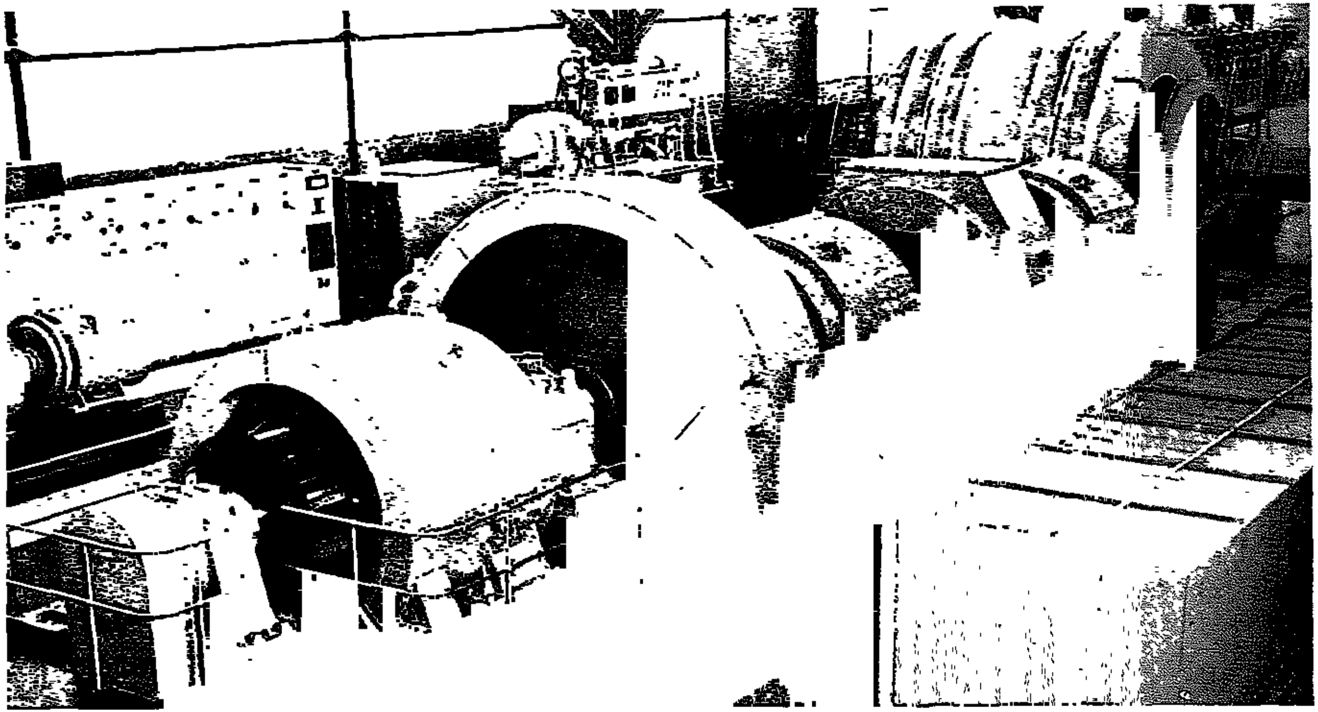


PLATE VIII ILGNER SET FOR ROLLING-MILL DRIVE

Note 54[illegible] reversing plate-mill motor in background

(Courtesy English Electric Co. Ltd.)

Now $$\phi_{SY} = \frac{\phi_{RY}}{1 + \sigma_r} \quad . \quad . \quad . \quad (4.109)$$

ϕ_{RY} can be obtained in the usual way by solving the second-order differential equation (4.106). The solution is given in equation (4.84).

It should be noted that

$$\sigma_s = RP_s = \frac{\text{leakage flux in stator}}{\text{useful flux in stator}}$$

and $$\sigma_r = RP_r = \frac{\text{leakage flux in rotor}}{\text{useful flux in rotor}}$$

Both σ_s and σ_r are small, so their products and squares can be neglected.

Under this assumption,

$$\phi_{SY} \simeq \frac{\omega\hat{\phi}_{SX} \sin(\omega_0 t - \lambda)}{\sqrt{\left\{(\sigma_r + \sigma_s)(\omega^2 - \omega_0^2)\frac{k_2}{R} + \frac{R(1 + \sigma_s)}{k_2}\right\}^2 + [1 + 2\sigma_r + 2\sigma_s]^2\omega_0^2}} \quad . \quad . \quad (4.110)$$

and $$\tan\lambda \simeq \frac{(1 + 2\sigma_r + 2\sigma_s)\omega_0}{(\sigma_r + \sigma_s)(\omega^2 - \omega_0^2)\frac{k_2}{R} + \frac{R}{k_2}(1 + \sigma_s)} \quad . \quad (4.111)$$

In order to predict the performance of the motor, we need to determine the magneto-motive force in the X-axis of the rotor and also the m.m.f. in the Y-axis. From the m.m.f.s we shall be able to determine the stator and rotor currents. All these quantities will be determined in terms of ϕ_{SX}, which is determined also from the applied voltage and frequency.

Determination of the m.m.f. in the X-axis of the Rotor, F_{RX}

$$F_{RX} = -k_2\omega\phi_{RY} - k_2\phi_{RX}' \quad . \quad . \quad . \quad (4.112)$$

$$= -k_2\omega\phi_{RY} - k_2\left[\frac{F_{RY}' + k_2\phi_{RY}''}{k_2\omega}\right]$$

$$= -k_2\omega\phi_{RY} - \frac{R\phi_{SY}'}{\omega} - \frac{k_2\phi_{RY}''}{\omega} \quad . \quad . \quad (4.113)$$

since $F_{RY}' = \phi_{SY}'R$.

$$\therefore\ F_{RX} = -k_2\omega(1 + \sigma_r)\phi_{SY} - \frac{R\phi_{SY}'}{\omega} - \frac{k_2(1 + \sigma_r)\phi_{SY}''}{\omega} \quad . \quad (4.114)$$

$$= \left[-k_2\omega(1 + \sigma_r) - \frac{RD}{\omega} - \frac{k_2(1 + \sigma_r)D^2}{\omega}\right]\phi_{SY} \quad . \quad (4.115)$$

Equation (4.115) may also be expressed thus—

$$F_{RX} = [-k_2\omega^2(1 + \sigma_r) - RD - k_2(1 + \sigma_r)D^2]\frac{\phi_{SY}}{\omega} \quad (4.116)$$

where for ϕ_{SY} is substituted its value given by equation (4.110). The rotor current is determined from the m.m.f.

Determination of F_{SX}, the m.m.f. in the X-axis of the Stator

We have
$$\phi_{SX} = \frac{F_{SX} + F_{RX}}{R} + F_{SX}P_{SX} \quad . \quad . \quad (4.117)$$

i.e. the total flux in the X-axis of the stator is equal to the flux in the X-axis, which crosses the air-gap, plus the stator leakage flux. F_{RX} will be negative in any numerical example. It is the component of the rotor m.m.f. measured in the *positive* direction of the X-axis.

$$F_{SX} = \frac{(R\phi_{SX} - F_{RX})}{1 + \sigma_s} \quad . \quad . \quad . \quad (4.118)$$

since $RP_{SX} = \sigma_s$

but

$$\phi_{SX} = \frac{\left[\left\{(1+\sigma_r)(\sigma_r+\sigma_r\sigma_s+\sigma_s)\frac{k_2}{R}\right\}\{\omega^2+D^2\} + D(1+2\sigma_r+2\sigma_s+2\sigma_r\sigma_s)+\frac{R}{k_2}(1+\sigma_s)\right]\phi_{SY}}{\omega} \quad (4.119)$$

and

$$F_{SX} = R\left[\frac{(\omega^2+D^2)(2\sigma_r+1)}{R}\times k_2 + 2D(1+\sigma_r)+\frac{R}{k_2}\right]\frac{\phi_{SY}}{\omega} \quad (4.120)$$

$$= R\left[\frac{(\omega^2+D^2)(2\sigma_r+1)}{R}k_2 + 2D(1+\sigma_r)+\frac{R}{k_2}\right]\times \frac{\hat{\phi}_{SX}\sin(\omega_0 t-\alpha)}{\sqrt{\left\{(\sigma_r+\sigma_s)(\omega^2-\omega_0^2)\frac{k_2}{R}+\frac{R(1+\sigma_s)}{k_2}\right\}^2+\{1+2\sigma_r+2\sigma_s\}^2\omega_0^2}} \quad . \quad . \quad (4.121)$$

where ϕ_{SY} is substituted in terms of ϕ_{SX} from equation (4.110).

Determination of ϕ_{SX} in Terms of the Applied Voltage per Phase

Let $v = \hat{V}\sin(\omega_0 t + \delta)$ = applied volts to stator per phase

N_1 = number of turns in the stator winding per phase in series

k_1 = breadth factor for the fundamental

k_3 = coil span factor for the fundamental

I = r.m.s. value of current in the stator

For a single-phase winding, the amplitude of the m.m.f. wave in ampere-turns

$$\hat{F}_{SX} = 1{\cdot}8N_1 \times k_1k_3 \times \frac{I}{p} \qquad (4{\cdot}122)$$

where p = number of poles

Now F_{SX} will be obtained in the form $a\hat{\phi}_{SX} \sin(\omega_0 t + \beta)$

$$\therefore\ v = N_1k_1k_3\omega_0\hat{\phi}_{SX}\frac{\cos\omega_0 t}{10^8} + \frac{(R_S a\hat{\phi}_{SX}\sin(\omega_0 t + \beta))p}{1{\cdot}8N_1k_1k_3} \qquad (4{\cdot}123)$$

R_s = stator resistance per phase.

Let
$$c = \frac{N_1k_1k_3\omega_0}{10^8} \qquad (4{\cdot}124)$$

and
$$d = \frac{R_s ap}{1{\cdot}8N_1 \times k_1 \times k_3} \qquad (4{\cdot}125)$$

then

$$v = c\hat{\phi}_{SX}\cos\omega_0 t + d\hat{\phi}_{SX}\sin(\omega_0 t + \beta) \qquad (4{\cdot}126)$$
$$= c\hat{\phi}_{SX}\cos\omega_0 t + d\hat{\phi}_{SX}\sin\omega_0 t\cos\beta + d\hat{\phi}_{SX}\cos\omega_0 t\sin\beta \qquad (4{\cdot}127)$$
$$= \hat{\phi}_{SX}[(c + d\sin\beta)\cos\omega_0 t + d\cos\beta\sin\omega_0 t] \qquad (4{\cdot}128)$$
$$= \hat{\phi}_{SX}[f\cos\omega_0 t + g\sin\omega_0 t] \qquad (4{\cdot}129)$$

where
$$f = c + d\sin\beta \qquad (4{\cdot}130)$$

and
$$g = d\cos\beta$$

Then
$$v = \hat{\phi}_{SX}k\sin(\omega_0 t + \delta) \qquad (4{\cdot}131)$$

where
$$k\cos\delta = g \qquad (4{\cdot}132)$$
$$k\sin\delta = f \qquad (4{\cdot}133)$$
$$k = \sqrt{g^2 + f^2} \qquad (4{\cdot}134)$$

and
$$\tan\delta = \frac{f}{g} \qquad (4{\cdot}135)$$

$$\therefore\quad k = \sqrt{d^2\cos^2\beta + c^2 + 2cd\sin\beta + d^2\sin^2\beta} \qquad (4{\cdot}136)$$

and
$$\tan\delta = \frac{c + d\sin\beta}{d\cos\beta} \qquad (4{\cdot}137)$$

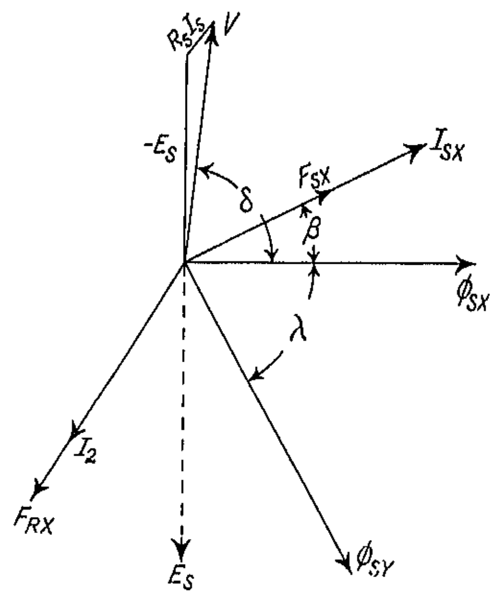
Now
$$\hat{V} = \hat{\phi}_{SX} \times k \qquad (4{\cdot}138)$$

$$\therefore\quad \hat{\phi}_{SX} = \frac{\hat{V}}{k} = \frac{\hat{V}}{\sqrt{c^2 + d^2 + 2cd\sin\beta}} \qquad (4{\cdot}139)$$

where $\hat{V}$ = maximum value of V

We thus have $\hat{\phi}_{SX}$ determined in terms of $\hat{V}$ and also its phase angle δ with respect to $\hat{V}$. All the other quantities are determined with respect to $\hat{\phi}_{SX}$. They are, therefore, known and their phase relations. The m.m.f. F_{SX} is known in terms of $\hat{\phi}_{SX}$ and its phase relation. The stator current is known, and its phase relation. ϕ_{RY} is known, and F_{RX} and F_{RY} are known, and, therefore, the rotor currents in the X- and Y-axes are known.

Since all quantities are known in terms of ϕ_{SX}, and ϕ_{SX} is known in terms of V, the whole characteristics of the motor are determined. We may now set out the vector diagram (Fig. 4.8).



ϕ_{SX} = flux in stator in the X-axis, which is the stator axis

F_{SX} = m.m.f. of stator

I_{SX} = current in stator

E_S = back e.m.f. in stator, 90° behind ϕ_{SX}

V = supply volts, and

F_{RX} = rotor m.m.f.

FIG. 4.8

The following test results on a three-phase machine running (*a*) as a three-phase machine, and (*b*) as a single-phase machine are of interest to the student.

CHARACTERISTICS OF A SINGLE-PHASE INDUCTION MOTOR

Test Machine. Brook motor. No. 38496.

220 V	Speed 1200 r.p.m.
10 h.p.	Cycles 60 c/s

Length of Prony brake arm = 30 in.

Effective weight of Prony brake arm = 4 lb.

Observations—

Three-phase Operation

Volts	Amperes	Kilowatts			P.F. Cos ϕ	R.p.m.	Load			Torque lb-ft	H.p.	Efficiency (%)	Losses Watts	Slip (%)
		W_1	W_2	$W_1 + W_2$			W	S	Net					
230	20·2	6·60	3·55	10·15	0·89	1150	22	4·5	21·5	53·0	11·70	86·0	1420	4·20
230	24·2	5·60	2·80	8·40	0·87	1160	18	4·2	17·8	44·0	9·75	86·5	1120	3·30
231	20·0	4·60	1·96	6·56	0·82	1170	14	3·9	14·1	35·0	7·76	88·3	760	2·50
232	16·4	3·75	1·25	5·00	0·75	1180	10	3·3	10·7	26·5	5·94	88·6	560	1·66
232	13·7	2·90	0·55	3·45	0·64	1187	6	2·8	7·2	18·0	4·04	87·3	430	1·08
233	10·6	2·06	0·00	2·06	0·50	1195	2	2·1	3·9	9·6	2·19	79·2	420	0·42
231	8·0	1·15	− 0·65	0·50	0·16	1200	0	0·0	4·0	9·8	2·20	0·0	—	0·00

Single-phase Operation

Volts	Amperes	Kilowatts	P.F. Cos ϕ	R.p.m.	Load			Torque lb-ft	H.p.	Efficiency (%)	Losses Watts	Slip (%)
					W	S	Net					
234	16·5	2·10	0·540	1190	2	2·25	3·75	9·26	2·1	71·6	530	0·83
232	22·3	3·70	0·715	1185	6	2·90	7·10	17·80	3·9	80·0	710	1·25
232	30·0	5·58	0·800	1181	10	3·60	10·40	25·70	5·8	77·2	1260	1·58
231	41·2	7·80	0·820	1150	14	4·10	13·90	34·30	7·5	71·8	2100	4·17
231	36·2	6·90	0·825	1155	12	3·60	12·40	30·60	6·7	73·0	1860	3·75
232	26·2	4·72	0·775	1170	8	3·10	8·90	22·00	4·9	77·5	1060	2·50
232	19·1	2·90	0·655	1185	4	2·50	5·50	13·60	3·1	79·0	610	1·25
232	13·2	0·40	—	1200	0	0·00	4·00	9·90	2·3	0·0	—	—

Calculations

Case 1

$$\text{Torque} = (30)(\tfrac{1}{12})(22 - 4{\cdot}5 + 4) \qquad \simeq 53 \text{ lb-ft}$$

$$\text{Total losses} = \text{input} - \text{output}$$

$$= 10\,150 - (11{\cdot}7)(746) \qquad \simeq 1420 \text{ W}$$

$$\text{Percentage efficiency} = \frac{\text{output}}{\text{input}} = \frac{(11{\cdot}7)(746)}{10\,150} \qquad = 86 \text{ per cent}$$

$$\text{Percentage slip} = \frac{\text{synchronous speed} - \text{actual speed}}{\text{synchronous speed}}$$

$$= \frac{1200 - 1150}{1200} \qquad = 4{\cdot}16 \text{ per cent}$$

From graph (Fig. 4.9)—

Losses when run as an induction motor (three-phase) at 10 h.p. $\simeq$ 1100 W

Single-phase output with the same loss (Fig. 4.10) = 5·1 h.p.

The characteristics of the motor under various conditions are shown in the graphs of Figs. 4.9, 4.10. and 4.11.

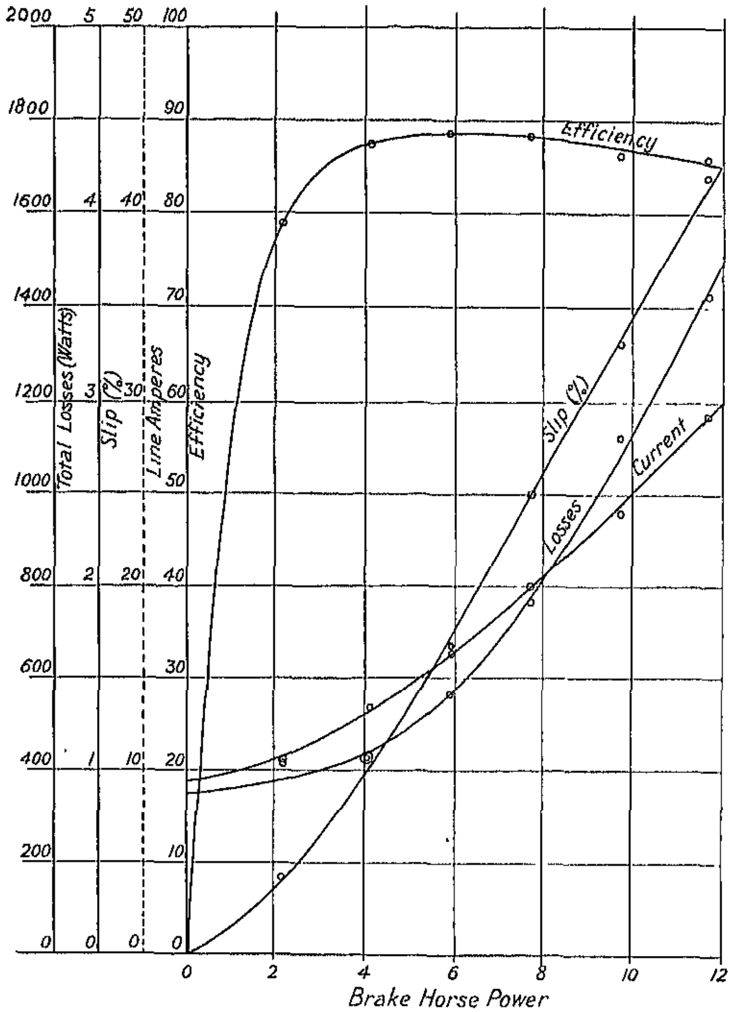


FIG. 4·9. CHARACTERISTICS OF A SINGLE-PHASE INDUCTION MOTOR ON THREE-PHASE OPERATION

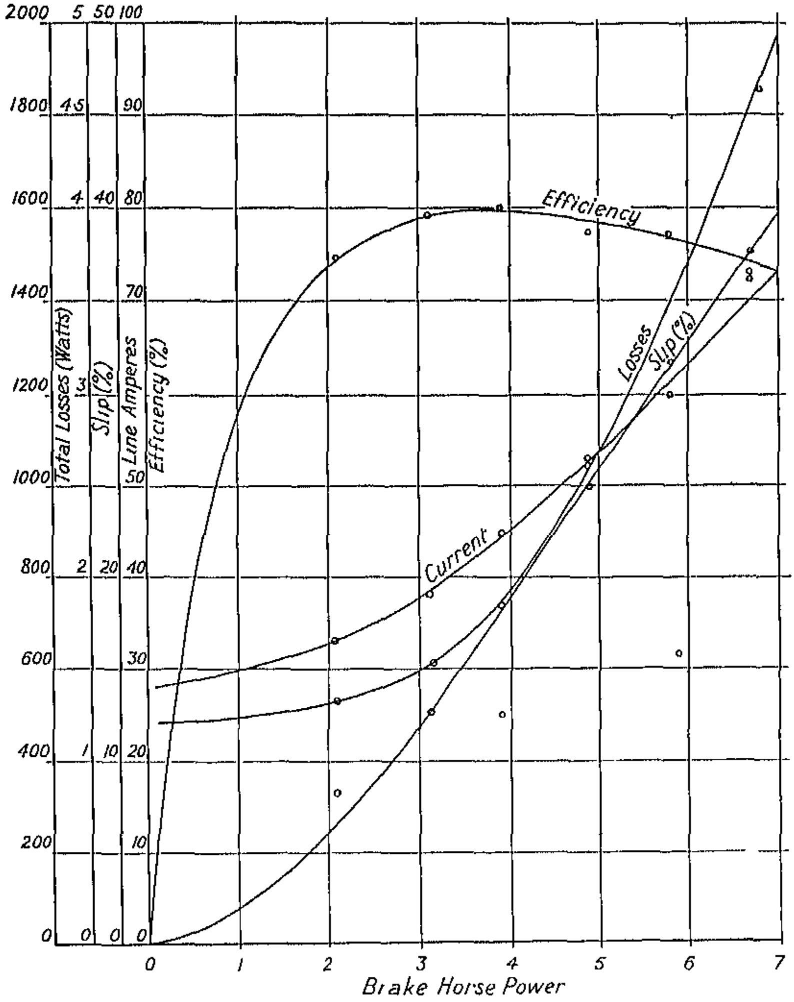


FIG. 4.10. CHARACTERISTICS OF A SINGLE-PHASE INDUCTION MOTOR ON SINGLE-PHASE OPERATION

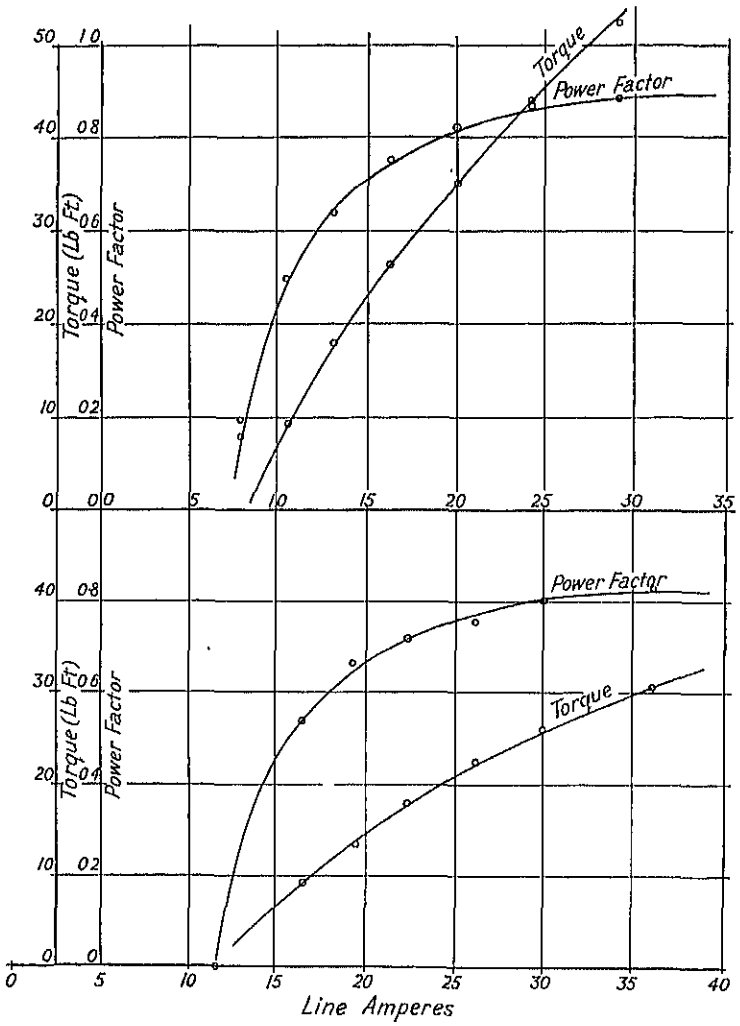


FIG. 4.11. TORQUE AND POWER-FACTOR CHARACTERISTICS FOR THE SINGLE-PHASE INDUCTION MOTOR OPERATED UNDER THREE-PHASE AND SINGLE-PHASE CONDITIONS

CHAPTER V

The Two Revolving Field Theory

This theory is based on the simple conception that a pulsating sinusoidal wave of flux or m.m.f. can be resolved into two rotating waves, each of half the amplitude of the pulsating wave, and rotating at synchronous speed in opposite directions.

Thus, consider a coil, or several coils, constituting a single-phase winding, and carrying alternating current of frequency f.

The m.m.f. curve of such a winding consists of several rectangular waves, displaced in space by the slot-pitch angle. The resultant wave, for the phase, is a stepped rectangular wave, which can be replaced by a series of sine waves of different frequency. Confining attention to the fundamental wave of flux density in the air-gap, due to the fundamental m.m.f., we have

$$B = a\sqrt{2}I \sin \omega t \sin x \qquad . \quad . \quad . \quad (5.1)$$

This is a wave distributed sinusoidally in space, as shown by the factor sin x, and the maximum value, in space, varies, also sinusoidally, with time, as shown by the factor sin ωt. This curve remains stationary in space, but its values over the pole pitch pulsate, but still remain sinusoidal in space.

Now $$\sin \omega t \sin x = \tfrac{1}{2}[\cos(x - \omega t) - \cos(x + \omega t)] \qquad . \quad (5.2)$$

$$\therefore \qquad B = a\frac{I}{\sqrt{2}}\,[\cos(x - \omega t) - \cos(x + \omega t)] \quad . \quad . \quad (5.3)$$

Equation (5.3) represents two rotating waves, each having half the amplitude of the pulsating wave. Each of these rotating waves will generate e.m.f.s in the rotor. At standstill the e.m.f.s will produce currents in the rotor which produce equal driving and retarding torques, so there is no starting torque. If the rotor is given rotation in any given direction it will continue to run in that direction. If ω is the angular velocity of the rotor, and ω_0 that of the rotating fields, then the relative angular velocity of the field and the rotor, for one wave, is $\omega_0 - \omega$ and the relative angular velocity of the second rotating field with respect to the rotor $= \omega_0 + \omega$.

The slip of the rotor with respect to the field running in the same direction as the rotor

$$= s = \frac{\omega_0 - \omega}{\omega_0} \qquad . \quad . \quad . \quad . \quad (5.4)$$

and $$\omega = \omega_0(1 - s) \qquad . \quad . \quad . \quad . \quad (5.5)$$

The slip of the rotor with respect to the field which runs in the opposite direction to that of the rotor

$$= \frac{\omega_0 + \omega}{\omega_0} = \frac{\omega_0 + \omega_0(1 - s)}{\omega_0}$$

$$= 2 - s \qquad . \quad . \quad . \quad . \quad . \quad . \quad (5.6)$$

Clearly the slip of the rotor, with regard to the reverse field, is nearly 2 for the normal range of operation of the motor. The resistance corresponding to the rotor input in the equivalent circuit, namely $\frac{r_2}{s}$ for the forward field and $\frac{r_2}{2 - s}$ for the reverse field, shows at once that the rotor input of the motor, for the reverse field, is small compared to that for the forward field, and the torque is a retarding torque due to it.

The behaviour of the motor is similar, *in some respects*, to two polyphase machines, whose stator windings are in series, and which have the same rotor, but the connections of the second motor are made in such a manner that the fields rotate in opposite directions in the two machines. This representation is not true to the facts, for in a polyphase machine, the torque is a constant torque, whereas in the single-phase machine, the torque is a pulsating torque. It will be clear also that the forward rotating field absorbs the larger share of the supply volts—in fact, much the larger share. Since the slip for the reverse field is $2 - s$, it follows that, for this field, the machine acts as if it were on short-circuit. The effect of this field can be taken into account by an *increase* in the primary leakage of the motor.

Let $\boldsymbol{V}$ = applied volts = vector of reference = V (Fig. 5.1)

r_1 = stator resistance of running winding

x_1 = stator *leakage* reactance at supply frequency

$\boldsymbol{I}_1$ = stator current

$\boldsymbol{I}_2'$ = rotor current, referred to the stator, due to forward rotating field, i.e. the field which rotates in the same direction as the rotor

$\boldsymbol{I}_{2R}'$ = rotor current, referred to the stator, due to the reverse field

x_m = reactance of the magnetizing circuit for each field at full supply frequency; note *this is one half the reactance of the pulsating field*

Then

$$\boldsymbol{V} = (r_1 + jx_1)\boldsymbol{I}_1 + jx_m(\boldsymbol{I}_2' + \boldsymbol{I}_1) + jx_m(\boldsymbol{I}_{2R}' + \boldsymbol{I}_1) \quad . \quad (5.7)$$

This is the equation for the stator.

For the rotor, we have

$$\left(\frac{r_2'}{s} + jx_2'\right) I_2' + jx_m(I_1 + I_2') = 0 \quad . \quad . \quad (5.8)$$

and
$$\left(\frac{r_2'}{2 - s} + jx_2'\right) I_{2R}' + jx_m(I_1 + I_{2R}') = 0 \quad . \quad . \quad (5.9)$$

where r_2' and x_2' are the rotor resistance and reactance, referred to the stator, *divided by* 2.

Equations (5.8) and (5.9) refer to the rotor circuit voltages for the forward and reverse fields respectively.

From equation (5.8) we can obtain I_2' in terms of I_1 and from equation (5.9) we can obtain I_{2R}' in terms of I_1.

If we substitute the values, so obtained, for these quantities in equation (5.7), we have a relation between V and I_1, from which we

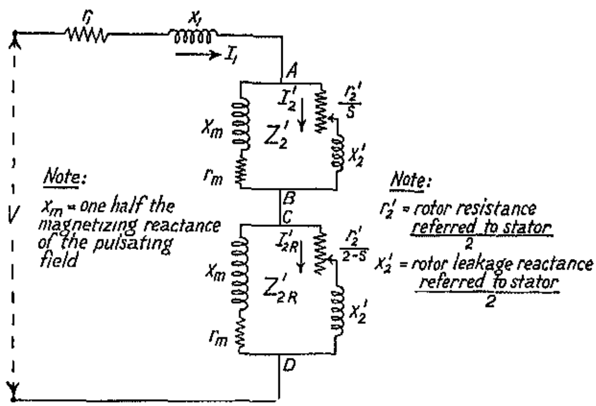


FIG. 5.1

can determine I_1 and its phase relation with respect to V. Then from equations (5.8) and (5.9) we can also obtain I_2' and I_{2R}' in terms of I_1 and hence in terms of V, and also their phase angles with V.

Inspection of equations (5.7), (5.8), and (5.9) show that the equivalent circuit for the single-phase motor is as shown in Fig. 5.1.

From equation (5.8) we have, neglecting r_m,

$$I_2' = \frac{-jx_m I_1}{\frac{r_2'}{s} + j(x_2' + x_m)} \quad . \quad . \quad . \quad (5.10)$$

and from equation (5.9), we have,

$$I_{2R}' = \frac{-jx_m I_1}{\frac{r_2'}{2 - s} + j(x_2' + x_m)} \quad . \quad . \quad . \quad (5.11)$$

Therefore, equation (5.7) becomes

$$V = (r_1 + jx_1)I_1 + jx_mI_1 + \frac{x_m^2 I_1}{\frac{r_2'}{s} + j(x_2' + x_m)} + jx_mI_1 + \frac{x_m^2 I_1}{\frac{r_2'}{2-s} + j(x_2' + x_m)} \quad (5.12)$$

$$\therefore V = I_1\left[r_1 + x_m^2\left\{\frac{1}{\frac{r_2'}{s} + j(x_2' + x_m)} + \frac{1}{\frac{r_2'}{2-s} + j(x_2' + x_m)}\right\} + j(x_1 + 2x_m)\right] \quad (5.13)$$

$$= I_1\left[r_1 + \frac{x_m^2\frac{r_2'}{s}}{\left(\frac{r_2'}{s}\right)^2 + (x_2' + x_m)^2} - \frac{jx_m^2 \times (x_2' + x_m)}{\left(\frac{r_2'}{s}\right)^2 + (x_2' + x_m)^2} + \frac{x_m^2 \times \frac{r_2'}{2-s}}{\left(\frac{r_2'}{2-s}\right)^2 + (x_2' + x_m)^2} - \frac{jx_m^2(x_2' + x_m)}{\left(\frac{r_2'}{2-s}\right)^2 + (x_2' + x_m)^2} + j(x_1 + 2x_m)\right] \quad (5.14)$$

$$\therefore \qquad V = I_1[c + jd] \quad (5.15)$$

$$I_1 = \frac{V}{\sqrt{c^2 + d^2}} \quad (5.16)$$

where
$$c = r_1 + \frac{x_m^2\frac{r_2'}{s}}{\left(\frac{r_2'}{s}\right)^2 + (x_2' + x_m)^2} + \frac{x_m^2\frac{r_2'}{2-s}}{\left(\frac{r_2'}{2-s}\right)^2 + (x_2' + x_m)^2} \quad (5.17)$$

and
$$d = x_1 + 2x_m - \frac{x_m^2(x_2' + x_m)}{\left(\frac{r_2'}{s}\right)^2 + (x_2' + x_m)^2} - \frac{x_m^2(x_2' + x_m)}{\left(\frac{r_2'}{2-s}\right)^2 + (x_2' + x_m)^2} \quad (5.18)$$

and
$$\tan\phi_1{}^* = \frac{d}{c} \quad (5.19)$$

From equation (5.10)

$$I_2' = \frac{-jx_mI_1}{\frac{r_2'}{s} + j(x_2' + x_m)} \quad (5.20)$$

* ϕ_1 = angle of lag of I_1 behind V

$$= \frac{-jx_m}{\frac{r_2'}{s} + j(x_2' + x_m)} \times \frac{V}{(c + jd)} \quad . \quad . \quad . \quad . \quad (5.21)$$

$$= \frac{-Vjx_m}{\frac{r_2'c}{s} - d(x_2' + x_m) + j\left\{\frac{dr_2'}{s} + c(x_2' + x_m)\right\}} \quad . \quad . \quad (5.22)$$

$$I_2' = \frac{Vx_m}{-\left\{\frac{dr_2'}{s} + c(x_2' + x_m)\right\} + j\left\{\frac{r_2'c}{s} - d(x_2' + x_m)\right\}} \quad . \quad (5.23)$$

and $$I_2' = \frac{Vx_m}{\sqrt{\left\{\frac{dr_2'}{s} + c(x_2' + x_m)\right\}^2 + \left\{\frac{r_2{}^2c}{s} - d(x_2' + x_m)\right\}^2}} \quad . \quad (5.24)$$

Also $$I_{2R}' = \frac{-jx_m I_1}{\frac{r_2'}{2-s} + j(x_2' + x_m)}$$

$$= \frac{-jx_m}{\frac{r_2'}{2-s} + j(x_2' + x_m)} \times \frac{V}{(c + jd)} \quad . \quad . \quad . \quad (5.25)$$

$$= \frac{-jx_m \times V}{\frac{cr_2'}{2-s} - d(x_2' + x_m) + j\left\{c(x_2' + x_m) + \frac{dr_2'}{2-s}\right\}} \quad . \quad (5.26)$$

$$= \frac{x_m V}{-\left\{c(x_2' + x_m) + \frac{dr_2'}{2-s}\right\} + j\left\{\frac{cr_2'}{2-s} - d(x_2' + x_m)\right\}} \quad . \quad (5.27)$$

$$\therefore \quad I_{2R}' = \frac{x_m V}{\sqrt{\left\{c(x_2' + x_m) + \frac{dr_2'}{2-s}\right\}^2 + \left\{\frac{cr_2'}{2-s} - d(x_2' + x_m)\right\}^2}} \quad (5.28)$$

Note that when $s = 1$, i.e. at standstill

$$I_2' = I_{2R}'$$

and the torques due to the two currents are equal and opposite, since the field producing I_2' rotates in the opposite direction to that producing I_{2R}'. Thus, at standstill, the torques are equal and opposite and there is no starting torque.

When $s = 1$, neglecting iron loss, that is neglecting r_m

$$\mathbf{Z}_2' = \frac{(r_2' + jx_2')(jx_m)}{r_2' + j(x_2' + x_m)} = \frac{jr_2'x_m - x_2'x_m}{r_2' + j(x_2' + x_m)} \quad . \quad . \quad . \quad (5.29)$$

$$= \frac{-x_2'x_m + jr_2'x_m}{r_2' + j(x_2' + x_m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.30)$$

$$= \frac{-r_2'x_2'x_m + j(r_2')^2x_m + jx_2'x_m(x_2' + x_m) + r_2'x_m(x_2' + x_m)}{(r_2')^2 + (x_2' + x_m)^2} \quad (5.31)$$

and $\mathbf{Z}_{2R}'$ has the same value at $s = 1$.

So that at standstill

$$I_1 = \frac{V}{r_1 + jx_1 + 2\underset{s=1}{\mathbf{Z}_2'}} \quad . \quad . \quad . \quad (5.32)$$

Z_2' may be written $e + jf$,

where $$e = -\frac{r_2'x_2'x_m + r_2'x_m(x_2' + x_m)}{(r_2')^2 + (x_2' + x_m)^2} \quad . \quad . \quad (5.33)$$

and $$f = \frac{(r_2')^2x_m + x_2'x_m(x_2' + x_m)}{(r_2')^2 + (x_2' + x_m)^2} \quad . \quad . \quad (5.34)$$

and $$\underset{(s=1)}{I_1} = \frac{V}{r_1 + jx_1 + 2e + 2jf} \quad . \quad . \quad (5.35)$$

$$= \frac{V}{r_1 + 2e + j(x_1 + 2f)} \quad . \quad . \quad (5.36)$$

$$\underset{(s=1)}{\bar{I}_1} = \frac{\bar{V}}{\sqrt{(r_1 + 2e)^2 + (x_1 + 2f)^2}} \quad . \quad . \quad (5.37)$$

and $$\tan \phi_{sc} = \frac{x_1 + 2f}{r_1 + 2e} \quad . \quad . \quad . \quad (5.38)$$

Equation (5.37) gives the short-circuit current and equation (5.38) its phase angle with respect to V.

Now look at Fig. 5.1 and note that Z_2' and Z_{2R}' change values when $s = 0$ and when $s = 2$, so that the primary current is the same when $s = 0$ and when $s = 2$. When $s = 0$ or $s = 2$, by substituting in equation (5.14), we get the stator current and its power factor.

Thus, $s = 0$ or $s = 2$,

$$V = (r_1 + jx_1)I_1 + 2jx_mI_1 + \frac{x_m^2 \dfrac{r_2'}{2} I_1 - jx_m^2(x_2' + x_m)I_1}{\left(\dfrac{r_2'}{2}\right)^2 + (x_2' + x_m)^2} \quad . \quad (5.39)$$

At $s = 0$ or $s = 2$,

$$V = I_1 \left\{ \left[r_1 + \frac{x_m^2 \frac{r_2'}{2}}{\left(\frac{r_2'}{2}\right)^2 + (x_2' + x_m)^2} \right] + j \left[2x_m + x_1 - \frac{x_m^2(x_2' + x_m)}{\left(\frac{r_2'}{2}\right)^2 + (x_2' + x_m)^2} \right] \right\} \quad (5.40)$$

$$= I_1(g + jh) \quad (5.41)$$

where
$$g = r_1 + \frac{x_m^2 \frac{r_2'}{2}}{\left(\frac{r_2'}{2}\right)^2 + (x_2' + x_m)^2} \quad (5.42)$$

and
$$h = 2x_m - \frac{x_m^2(x_2' + x_m)}{\left(\frac{r_2'}{2}\right)^2 + (x_2' + x_m)^2} + x_1 \quad (5.43)$$

$$\therefore \qquad \underset{(s=0 \text{ or } s=2)}{I_1} = \frac{V}{\sqrt{g^2 + h^2}} \quad (5.44)$$

and
$$\underset{(s=0 \text{ or } s=2)}{\tan \phi} = \frac{h}{g} \quad (5.45)$$

With the rotor circuit open at $s = 0$,

$$I_1 = \frac{V}{\sqrt{r_1 + (x_1 + 2x_m)^2}} \quad (5.46)$$

The ratio of equations (5.46) to (5.44), i.e. the

$$\frac{\text{primary current when rotor is open when } s = 0}{\text{primary current when the rotor is closed}} \simeq \frac{1}{2}$$

The value of I_1 when $s = \infty$,

$$V = \underset{(s=\infty)}{I_1} \left[r_1 - \frac{jx_m^2}{x_2' + x_m} - \frac{jx_m^2}{x_2' + x_m} + j(x_1 + 2x_m) \right] \quad (5.47)$$

$$\therefore \qquad \underset{(s=\infty)}{I_1} = \frac{V}{r_1 - \frac{2jx_m^2}{x_2' + x_m} + j(x_1 + 2x_m)} \quad (5.48)$$

and
$$\underset{(s=\infty)}{I_1} = \frac{V}{\sqrt{r_1^2 + \left(x_1 + 2x_m - \frac{2x_m^2}{x_2' + x_m}\right)^2}} \quad (5.49)$$

and
$$\tan\phi_{so} = \frac{x_1 + 2x_m - \dfrac{2x_m^2}{x_2' + x_m}}{r_1} \qquad . \qquad . \quad (5\cdot50)$$

Thus, we have determined points on the current locus for $s = 0$, $s = \infty$, $s = 1$, and $s = 2$ and the phase angles made by the current, at these points, with respect to the applied volts.

By the Principle of Inversion, our equation (5.14) can be shown to represent a circle, for it is of the form

$$\frac{\mathbf{I}_1}{\mathbf{V}_1} = \frac{C + Ds(2-s)}{E + Fs(2-s)} \qquad . \qquad . \qquad . \quad (5\cdot51)$$

where C, D, E, and F are complex constants and s = slip of the motor with respect to the forward field.

We may obtain all the information we require about the performance of our motor from the equivalent circuit shown in Fig. 5.1.

Thus, the torque in synchronous watts = input to the rotor. The torque in synchronous watts

$$= (I_2')^2 \frac{r_2'}{s} - (I_{2R}')^2 \frac{r_2'}{2-s} \qquad . \qquad . \qquad . \quad (5\cdot52)$$

The term $(I_2')^2 \dfrac{r_2'}{s}$ represents the forward torque in synchronous watts and $(I_{2R}')^2 \dfrac{r_2'}{2-s}$ represents the backward torque due to the reverse field.

The torque in lb-ft

$$= \frac{7\cdot04}{\text{r.p.m.}_{\text{syn}}} \left\{ (I_2')^2 \frac{r_2'}{s} - (I_{2R}')^2 \frac{r_2'}{2-s} \right\} \qquad . \qquad . \quad (5\cdot53)$$

where $\text{r.p.m.}_{\text{syn}}$ = synchronous speed in revolutions per minute.

The gross mechanical power in horse-power

$$= \frac{2\pi\, \text{r.p.m.}_{\text{syn}}(1-s) \times 7\cdot04}{\text{r.p.m.}_{\text{syn}} \times 33\,000} \left\{ (I_2')^2 \frac{r_2'}{s} - (I_{2R}')^2 \frac{r_2'}{2-s} \right\} \quad (5\cdot54)$$

$$= \frac{(1-s)}{746} \left\{ (I_2')^2 \frac{r_2'}{s} - (I_{2R}')^2 \frac{r_2'}{2-s} \right\} \text{h.p.} \qquad . \qquad . \quad (5\cdot55)$$

It is interesting to note that the copper loss in the rotor, due to I_{2R}', viz. $(I_{2R}')^2 r_2'$ is greater than the electrical input to the rotor from the reverse field. It follows that part of the copper loss must be supplied mechanically, i.e. the difference between $(I_{2R}')^2 r_2'$ and $(I_{2R}')^2 \dfrac{r_2'}{2-s}$ must be supplied mechanically through the shaft.

It will be clear, from inspection of Fig. 5.1, that the impedance Z_2', for the normal range of operation of the motor, is very much

greater than Z_{2R}', so that most of the voltage is impressed across Z_2', and in fact usually only about 10 per cent of the voltage is impressed on $\mathbf{Z}_{2R}'$. Also the impedance of $\mathbf{Z}_{2R}'$ is nearly equal to $\frac{r_2'}{2-s}+jx_2'$ for normal values of the slip, for

$$\mathbf{Z}_{2R}' = \frac{\left(\frac{r_2'}{2-s}+jx_2'\right)(jx_m)}{\frac{r_2'}{2-s}+j(x_2'+x_m)} \quad . \quad . \quad (5.56)$$

$$= \frac{\frac{r_2'}{2-s}+jx_2'}{\frac{r_2'}{j(x_m)(2-s)}+\frac{x_2'+x_m}{x_m}} \quad . \quad . \quad (5.57)$$

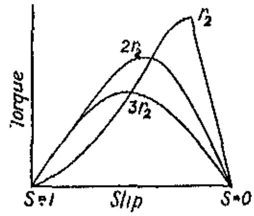


FIG. 5.2. TORQUE/SLIP CURVE FOR DIFFERENT VALUES OF ROTOR RESISTANCE

and since x_m is much greater than $\frac{r_2'}{2-s}$ and also much greater than x_2', it follows that $\mathbf{Z}_{2R}'$ is nearly equal to $\frac{r_2'}{2-s}+jx_2'$, so it follows that the performance of the motor in the usual range of operation can be determined approximately by increasing the leakage impedance of the stator by the amount $\frac{r_2'}{2-s}+jx_2'$.

Effect of Secondary Resistance and Reactance on the Torque/Speed Curve

In the polyphase motor the maximum torque is independent of the rotor resistance, which is only effective in determining the slip at which maximum torque occurs. In the single-phase motor, the secondary resistance not only determines the slip for maximum torque, but also determines its magnitude (*see* Fig. 5.2).

This can be clearly seen by inspecting equation (5.53), which gives us the torque in synchronous watts,

$$= \left\{(I_2')^2\frac{r_2'}{s} - (I_{2R}')^2\frac{r_2'}{2-s}\right\}$$

Now I_2' is given by equation (5.23) and I_{2R}' by equation (5.28).

If these values are substituted, and then the expression differentiated with respect to s and equated to zero, we shall get the slip for maximum torque, and then by substitution of this slip, we shall obtain the maximum torque. We see that the maximum torque is reduced by increasing the secondary resistance. Also at low values of the slip, the value of $\mathbf{Z}_{2R}'$ is nearly x_2', and this is in series with the forward impedance and reduces the maximum torque.

Iron Loss in Single-phase Motors

In Fig. 5.1 these losses are represented by a resistance r_m inserted in series with the magnetizing reactance x_m. If the impedance of the magnetizing circuit is represented by $Z_m = r_m + jx_m$, then the admittance of this circuit

$$= Y_m = \frac{1}{r_m + jx_m} = \frac{r_m}{r_m^2 + x_m^2} - \frac{jx_m}{r_m^2 + x_m^2} = g_m - jb_m \quad (5.58)$$

where g_m = conductance of the magnetizing circuit
and b_m = susceptance of the magnetizing circuit

$$r_m = \frac{g_m}{g_m^2 + b_m^2} \simeq \frac{g_m}{b_m^2} \text{ and } x_m = \frac{b_m}{g_m^2 + b_m^2} \simeq \frac{1}{b_m} \quad . \quad (5.59)$$

The current in the magnetizing circuits

$$= \text{volts across } Z_m \times (g_m - jb_m)$$

If W_i = iron loss due to the flux,

$$g_m = \frac{W_i}{E^2}$$

where E = r.m.s. volts across Z_m

There are, of course, iron losses due to each of the rotating fields, so it will be necessary to calculate the voltages across the two magnetizing circuits in Fig. 5.1.

We have already obtained expressions for I_2' and I_{2R}', so these voltages are

$$I_2' \sqrt{\left(\frac{r_2'}{s}\right)^2 + (x_2')^2} \text{ and } (I_{2R}')^2 \sqrt{\left(\frac{r_2'}{2-s}\right)^2 + (x_2')^2}$$

The iron losses are determined from the flux densities in the various parts of the machines, and, of course, these are determined by the fluxes and areas of the parts. The fluxes are determined from the voltages across $\mathbf{Z}_2'$ and $\mathbf{Z}_{2R}'$ in Fig. 5.1. Again iron-loss curves, appropriate to the frequency, must be used. The iron loss in the rotor must also be evaluated, and this is fairly large due to the reverse field. The current at zero slip is given in equation (5.44). If the no-load current is measured, it is possible to obtain an approximate expression for x_m. Otherwise, it may be obtained from the expression

$$x_m = 3\cdot2 \times \frac{\tau \times L}{\delta} \times K^2 \times \frac{N^2}{p} \times f \times 10^{-8} \text{ ohm} \quad . \quad (5.60)$$

where τ = pole pitch in centimetres
L = core length in centimetres
δ = air-gap length in centimetres, corrected for slots and saturation

K = product of breadth factor and coil-span factor for the stator winding

N = total number of turns on the stator for running phase

f = supply frequency

and p = poles

Referred Quantities

In Fig. 5.1. *all quantities are referred to the stator.*

If r_2 = actual resistance of the rotor

r_2' = rotor resistance, *referred to the stator*

x_2 = rotor reactance at supply frequency

x_2' = rotor reactance, at supply frequency, *referred to the stator*

and m_2 = number of rotor phases

Then
$$r_2' = \frac{1}{m_2}\left(\frac{N_1K_1}{N_2K_2}\right)^2 \times r_2 \quad . \quad . \quad . \quad (5.61)$$

where N_1 = turns per phase in stator

K_1 = winding factor for stator = breadth factor × coil-span factor

and N_2 = number of turns per phase in rotor

Also
$$x_2' = \frac{1}{m_2}\left(\frac{N_1K_1}{N_2K_2}\right)^2 x_2 \quad . \quad . \quad . \quad (5.62)$$

where K_2 = winding factor for the rotor

For the squirrel-cage rotor

$$m_2 = \frac{\text{number of rotor bars}}{\text{pairs of poles}}$$

and
$$N_2 = \tfrac{1}{2}; \quad K_2 = 1$$

$$\underset{\text{(referred current)}}{I_2'} = I_2 \times \frac{m_2N_2K_2}{N_1 \times K_1} \quad . \quad . \quad . \quad (5.63)$$

$$\underset{\text{(referred current)}}{I_{2R}'} = I_{2R} \times \frac{m_2N_2K_2}{N_1K_1} \quad . \quad . \quad (5.64)$$

All the quantities in Fig. 5.1, can now be determined. Given an actual motor, r_m and x_m can be determined from the no-load test.

It has been demonstrated that the single-phase motor, unaided, has no starting torque. There are several methods adopted to make it useful as a motor.

The most common method is to use a second phase, known as the starting phase, and to ensure, as far as possible, that this starting phase shall have quadrature relation in space, i.e. it shall be displaced

by 90° from the running phase, and further that the currents in the running and starting phases shall be displaced in time by 90°, or as near 90° as possible. If true quadrature relations are maintained in *both space* and *time* for the two phases, and if also the ampere-turns for the two phases are equal, then a true rotating field will be produced of constant amplitude, and constant speed. Since both running and starting windings are fed from a single-phase supply source, some form of phase splitting device is necessary. This can be done in various ways.

In one very common method used, the running phase occupies two-thirds of the slots per pole, and the starting phase the remaining one-third of the slots. The turns per coil in the starting phase are made double those for the running phase. This gives the same number of turns per phase for both phases, but the reactance of the starting phase is double that of the running phase, for the reactance

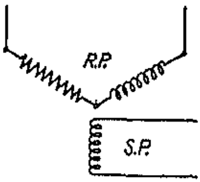


Fig. 5.3

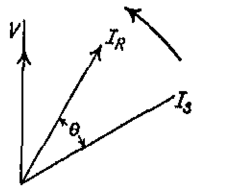


Fig. 5.4

of each coil, for the starting phase, is four times that of each coil in the running phase, but there are only half the number of coils in the starting phase, and so the reactance is doubled. One tries to make the reactance of one phase high and the reactance of the other phase low, and by adding resistance to the low reactance phase, one approximates to the condition for a time phase displacement of the currents sufficiently great to produce the starting torque required. If there is inequality of ampere-turns in both phases, and a phase displacement of less than 90° in time, then an elliptical field will be produced, rotating at a variable speed. For the analysis of such a motor, the methods of symmetrical components is necessary.

The starting torque in such a machine will be proportional to the product of the fluxes produced by the two phases multiplied by the sine of the angle between them. This method of phase splitting is only used in those cases where the starting conditions are easy, i.e. light load or no-load.

Sometimes a three-phase winding is used, with two of the phases connected in series for the running phase. The third phase is used for the starting phase. This method is shown in Fig. 5.3.

Fig. 5.4 shows the relation of the currents in the running phase I_R and in the starting phase I_S.

In the fractional horse-power field, split-phase motors are used from about $\frac{1}{20}$ to $\frac{1}{3}$ h.p., and usually for speeds of 2890, 1430, and 960 r.p.m. for 50 c/s, and 1425 r.p.m. for 25 c/s.

Capacitator-start Motors

In all these machines a squirrel-cage rotor is used and a capacitor is used in the starting phase, as shown in Fig. 5.5.

The relation of the currents in the phases to the supply volts is shown in Fig. 5.6. The current in the running winding lags the supply volts by the angle ϕ_1; the current in the starting phase leads V by ϕ_2. Then $\phi_1 + \phi_2 = 90°$ by adjustment.

This method is used in the smaller sizes of from $\frac{1}{8}$ to $\frac{3}{4}$ h.p., and up to 10 h.p. A centrifugal switch is usually fitted, cutting out the starting phase when up to speed.

With this method of starting torques up to 350 per cent of full-load torque are obtained, with a pull-out torque as single-phase

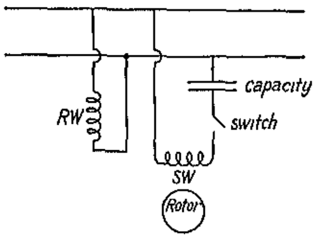


FIG. 5.5

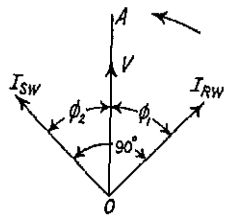


FIG. 5.6

motor of 250 per cent. The size of capacitance required varies with the output, being about 80 μF for $\frac{1}{8}$ h.p. to about 350 μF for 1 h.p.

In some cases, the motor runs with the second phase and capacitor permanently in circuit, i.e. as a two-phase unbalanced motor. This is the case in machines operating dental lathes, and in other applications. In this case a small capacitance is necessary for good operation, varying from 3 to 20 μF. The starting torque is only about 50 per cent of full-load torque with these low capacitances.

The electrolytic capacitor is used for starting purposes, but for running purposes, paper capacitors are used. These, oil insulated, paper capacitors are bigger and more expensive than the electrolytic types.

The Motor with Capacitance in Series with the Auxiliary Phase Vector Diagram

$V_M = OA =$ voltage across the main or running winding (Fig. 5.7)

$E_M = OC =$ voltage component to overcome the back e.m.f. in the main winding

$CB =$ voltage component to overcome the leakage reactance drop of the main winding

$BA =$ voltage component to overcome resistance drop in the main winding

$E_S = OD$ = voltage component to overcome back e.m.f. in starting winding

DE = voltage component to overcome resistance volts in starting winding

EF = voltage component to overcome leakage reactance drop in starting winding

FA = voltage component to overcome the capacitor reactive drop in starting winding

I_M = current in the main winding

I_S = current in starting winding, in quadrature with I_M

$V_S = OF$ = voltage across the starting winding

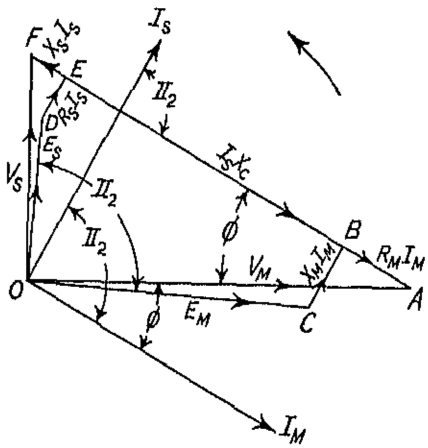


FIG. 5.7

The number of turns in the windings may not be equal, but the motor should act as a symmetrical two-phase machine. To that end the currents must be in time quadrature, and further the effective ampere-turns of each winding should be equal and in time quadrature. When that condition is satisfied the fluxes will be equal and in time quadrature, and it follows that the resistance and reactance of the auxiliary winding, when referred to the main winding, will be equal to those of the main winding.

Thus, r_s' = resistance of auxiliary winding referred to the main winding

$$= r_s \times \alpha^2 = r_m$$

If T_1 = number of turns in main winding

T_2 = number of turns in auxiliary winding

$f_1 = K_1 \times K_3$ = breadth factor × coil-span factor for main winding

$f_2 = K_2 \times K_4 =$ breadth factor $\times$ coil-span factor for auxiliary winding

$I_M =$ current in main winding

and $I_S =$ current in starting winding

$$I_M T_1 \times f_1 = I_S \times T_2 \times f_2$$

Then $$I_S = I_M \times \frac{T_1 \times f_1}{T_2 \times f_2} = I_M \times \alpha \qquad (5.65)$$

Since the ampere-turns are equal, the fluxes are equal and the back e.m.f.s of both windings are proportional to the number of effective turns,

i.e. $$\frac{E_S}{E_M} = \frac{T_2 \times f_2}{T_1 \times f_1} = \frac{1}{\alpha} \qquad (5.66)$$

$\therefore$ $$E_S = \frac{E_M}{\alpha} \qquad (5.67)$$

and, of course, $\bar{E}_S$ is in quadrature with $\bar{E}_M$.

Also $$x_s' = \alpha^2 x_s = x_m \qquad (5.68)$$

For this balanced condition, *neglecting the loss in the capacitor*,

$$\alpha = \frac{E_M}{E_S} \simeq \frac{V_M}{V_S} = \cot \phi \qquad (5.69)$$

for V_M and V_S are in quadrature.

Also $$VA \cos \phi = V_M = V \qquad (5.70)$$

$$I_S X_c \cos \phi = V$$

$$I_M X_c \alpha \cos \phi = V \qquad (5.71)$$

$\therefore$ $$X_c = \frac{V}{\alpha I_M \cos \phi} \qquad (5.72)$$

but $$X_c = \frac{1}{C\omega} \qquad (5.73)$$

$\therefore$ $$\frac{1}{C\omega} = \frac{V}{\alpha I_M \cos \phi} \qquad (5.74)$$

$\therefore$ $$\underset{\text{farads}}{C} = \frac{\alpha I_M \cos \phi}{\omega V} = \frac{\alpha I_M \cos \phi}{2\pi f V} \qquad (5.75)$$

Thus, the capacitance in farads, for an effective turns ratio α, is given by equation (5.75).

The current I_M is obtained on the basis of a symmetrical two-phase system,

i.e. $$I_M = \frac{\text{b.h.p.} \times 746}{2 \times V \times \cos \phi \times \text{efficiency}}$$

It is clear, from Fig. 5.7, that this ideal condition exists only for one load, preferably full load, and that any change in current, due to load changes, will upset the relation between V_S and V_M, for the line FA will alter in length.

Analysis of Operation of Capacitor Motor under Unbalanced Conditions

We have seen earlier that the m.m.f. of a single coil on the stator, carrying a sinusoidal current is rectangular in shape. We may replace this rectangular wave by its fundamental sine wave and its various harmonics. Confining attention to the fundamental wave, we saw that the flux density at any part of the pole pitch can be represented by

$$B_1 = a_1\sqrt{2}I_1 \sin \omega t \sin x \quad . \quad . \quad . \quad (5.1)$$

This is a pulsating wave of flux, and

a_1 = the factor which converts current to flux density

I_1 = the r.m.s. current in the coil

$\omega = 2\pi \times$ frequency

and x = distance measured from the coil side to the point, where the density is B

Now equation (5.1) is equivalent to

$$B_1 = \frac{a_1I_1}{\sqrt{2}}[\cos(x - \omega t) - \cos(x + \omega t)] \quad . \quad . \quad (5.3)$$

Equation (5.3) represents two rotating waves, one equal to the other, and each has half the amplitude of the pulsating wave. Let there be another set of coils, displaced in space by the angle β from the first set, and carrying a current differing in time phase by the angle λ.

Then keeping our origin the same, we have

$$B_2 = a_2\sqrt{2}I_2 \sin(\omega t - \lambda) \sin(x - \beta) \quad . \quad . \quad (5.76)$$

where B_2 is the density due to I_2 in the second set of coils at the point x.

Now

$$B_2 = \frac{a_2I_2}{\sqrt{2}}[\cos(x - \beta - \omega t + \lambda) - \cos(x - \beta + \omega t - \lambda)] \quad (5.77)$$

The second set of coils also produces two opposite rotating waves as given by (5.77).

Now make $\lambda = \pi - \beta$, then equation (5.77) becomes

$$B_2 = \frac{a_2I_2}{\sqrt{2}}[\cos(x - 2\beta - \omega t + \pi) - \cos(x + \omega t - \pi)]$$

$$= \frac{a_2I_2}{\sqrt{2}}[-\cos(x - 2\beta - \omega t) + \cos(x + \omega t)] \quad . \quad . \quad (5.78)$$

If $a_1I_1 = a_2I_2$, that is if the ampere-turns of the two sets of coils are equal, then it is clear that, when $\lambda = \pi - \beta$, the reverse fields cancel, and we have two positively rotating waves only of constant amplitude. Again, if $\lambda - \beta = \pi$, we have

$$B_2 = \frac{a_2I_2}{\sqrt{2}}[-\cos(x - \omega t) + \cos(x + \omega t - 2\beta)]$$

In this case the two coils produce a single rotating field, but rotating in the reverse direction.

In the first case, when $\lambda = \pi - \beta$, we have

$$B_1 + B_2 = \frac{a_1I_1}{\sqrt{2}}[\cos(x - \omega t) - \cos(x - 2\beta - \omega t)]$$

$$B = \frac{2a_1I_1}{\sqrt{2}}\sin(x - \omega t - \beta)\sin\beta \quad . \quad . \quad (5.79)$$

since we assumed $a_1I_1 = a_2I_2$.

Now our object in the analysis is to resolve our unbalanced currents and voltages into two symmetrical systems; one system of currents produce a field of constant amplitude and constant speed in one direction, called the "positive-sequence" system, and the other set, called the "negative-sequence" system, produces a field of constant amplitude and speed, revolving in the opposite direction.

Let T_1 = number of turns in the main winding

T_2 = number of turns in the starting winding

f_1 = breadth factor × coil-span factor for main winding

f_2 = breadth factor × coil-span factor for starting winding

Then $\dfrac{T_1 \times f_1}{T_2 \times f_2} = \alpha$

Let x_c = capacitive reactance of the capacitor in series with the starting winding

V = line volts

I_M = current in main winding

I_S = current in auxiliary winding

Then, since we assume that the ampere-turns of the two systems of currents in the two windings are equal, we have

$$I_M \times T_1 \times f_1 = I_S \times T_2 \times f_2$$

i.e.

$$I_M\alpha = I_S \quad . \quad . \quad . \quad . \quad (5.80)$$

If I_{M_1} and I_{M_2} represent the positive- and negative-sequence components of current in the main winding and I_{S_1} and I_{S_2} represent

the positive- and negative-sequence components of current in the auxiliary winding, we have

$$\mathbf{I}_M = \mathbf{I}_{M_1} + \mathbf{I}_{M_2} \quad . \quad . \quad . \quad . \quad (5.81)$$

$$\mathbf{I}_S = \mathbf{I}_{S_1} + \mathbf{I}_{S_2} \quad . \quad . \quad . \quad . \quad (5.82)$$

$$= +jI_{M_1}\alpha - jI_{M_2}\alpha \quad . \quad . \quad . \quad (5.83)$$

from which we have

$$\mathbf{I}_{M_1} = \frac{\mathbf{I}_M - j\dfrac{\mathbf{I}_S}{\alpha}}{2} \quad . \quad . \quad . \quad . \quad (5.84)$$

$$\mathbf{I}_{M_2} = \frac{\mathbf{I}_M + j\dfrac{\mathbf{I}_S}{\alpha}}{2} \quad . \quad . \quad . \quad (5.85)$$

Similarly

$$\mathbf{I}_{S_1} = \frac{\mathbf{I}_S + j\mathbf{I}_M\alpha}{2} \quad . \quad . \quad . \quad . \quad (5.86)$$

$$\mathbf{I}_{S_2} = \frac{\mathbf{I}_S - j\mathbf{I}_M\alpha}{2} \quad . \quad . \quad . \quad . \quad (5.87)$$

Let $\mathbf{E}_S$ and $\mathbf{E}_M$ be the actual generated e.m.f.s in the auxiliary and main windings respectively; these voltages may be resolved into symmetrical components also.

Thus $$\mathbf{E}_S = \mathbf{E}_{S_1} + \mathbf{E}_{S_2}$$

E_{S_1} = positive-sequence component

E_{S_2} = negative-sequence component

and
$$\mathbf{E}_M = \mathbf{E}_{M_1} + \mathbf{E}_{M_2}$$

$$= -j\mathbf{E}_{S_1}\alpha + j\mathbf{E}_{S_2}\alpha \quad . \quad . \quad . \quad (5.88)$$

$\therefore$
$$\mathbf{E}_{S_1} = \frac{\mathbf{E}_S + j\dfrac{\mathbf{E}_M}{\alpha}}{2} \quad . \quad . \quad . \quad . \quad (5.89)$$

and
$$\mathbf{E}_{S_2} = \frac{\mathbf{E}_S - j\dfrac{\mathbf{E}_M}{\alpha}}{2} \quad . \quad . \quad . \quad . \quad (5.90)$$

Also let
$$\mathbf{Z}_S = \mathbf{Z}_M \times \frac{1}{\alpha^2} \quad . \quad . \quad . \quad . \quad (5.91)$$

where $\mathbf{Z}_S$ = impedance of the auxiliary phase.

Now the positive-sequence components of the voltage, supplied

to the auxiliary phase—which has the capacitor in series with it—are given by the following equations—

$$\mathbf{V}_{S_1} = \mathbf{I}_{S_1}\left(-j\frac{x_c}{2} + Z_{S_1}\right) - j\frac{x_c}{2}\mathbf{I}_{S_2} \quad . \quad . \quad (5.92)$$

and

$$\mathbf{V}_{S_2} = \mathbf{I}_{S_2}\left(-j\frac{x_c}{2} + Z_{S_2}\right) - j\frac{x_c}{2}\mathbf{I}_{S_1} \quad . \quad . \quad (5.93)$$

The zero-order and second-order components of the capacitive reactance are both $-j\frac{x_c}{2}$

Also

$$V_{S_1} = \frac{V}{2}\left(1 + \frac{j}{\alpha}\right) \quad . \quad . \quad . \quad (5.94)$$

and

$$V_{S_2} = \frac{V}{2}\left(1 - \frac{j}{\alpha}\right) \quad . \quad . \quad . \quad (5.95)$$

$$\mathbf{I}_{S_1} = \frac{\frac{V}{2}\left\{\left(1 + \frac{j}{\alpha}\right) Z_{S_2} + \frac{x_c}{\alpha}\right\}}{\mathbf{Z}_{S_1}\mathbf{Z}_{S_2} - j\frac{x_c}{2}\left(\mathbf{Z}_{S_1} + \mathbf{Z}_{S_2}\right)} \quad . \quad . \quad (5.96)$$

and

$$\mathbf{I}_{S_2} = \frac{\frac{V}{2}\left\{\left(1 - \frac{j}{\alpha}\right) \mathbf{Z}_{S_1} - \frac{x_c}{\alpha}\right\}}{Z_{S_1}Z_{S_2} - j\frac{x_c}{2}\left(\mathbf{Z}_{S_1} + \mathbf{Z}_{S_2}\right)} \quad . \quad . \quad (5.97)$$

Also

$$\mathbf{I}_{M_1} = \frac{\frac{V}{2}\left\{\left(\frac{1}{\alpha^2} - \frac{j}{\alpha}\right) \mathbf{Z}_{M_2} - jx_c\right\}}{\frac{1}{\alpha^2}\mathbf{Z}_{m_1}\mathbf{Z}_{m_2} - j\frac{x_c}{2}\left(\mathbf{Z}_{m_1} + \mathbf{Z}_{m_2}\right)} \quad . \quad . \quad (5.98)$$

$$\mathbf{I}_{M_2} = \frac{\frac{V}{2}\left\{\left(\frac{1}{\alpha^2} + \frac{j}{\alpha}\right) \mathbf{Z}_{m_1} - jx_c\right\}}{\frac{1}{\alpha^2}\mathbf{Z}_{m_1}\mathbf{Z}_{m_2} - j\frac{x_c}{2}\left(\mathbf{Z}_{m_1} + \mathbf{Z}_{m_2}\right)} \quad . \quad . \quad (5.99)$$

where $\mathbf{Z}_{M_1}$ and $\mathbf{Z}_{M_2}$ represent the impedances of the main winding to positive- and negative-sequence currents. The starting torque is determined by finding the rotor currents corresponding to I_{M_1} and I_{M_2}, and referring both to the main winding of the stator. From the positive- and negative-sequence diagrams, we have the ratio—

$$\frac{I' \text{ rotor}}{I \text{ stator}} = \frac{\text{rotor current referred to stator}}{\text{stator current}} = \frac{+jx_m}{\frac{r_2'}{s} + j(x_m + x_2')}$$

Corresponding to I_{M_1}, we have the rotor current, referred to the stator—

$$I_{M_2}' = + I_{M_1} \times \frac{jx_m}{\frac{r_2'}{s} + j(x_m + x_2')} \quad . \quad . \quad (5.100)$$

I_{M_2} is the current in the rotor corresponding to I_{M_1}.

Likewise for the component I_{M_2} in the stator, we have

$$I_{MR_2}' = + I_{M_2} \frac{jx_m}{\frac{r_2'}{2-s} + j(x_m + x_2')} \quad . \quad . \quad (5.101)$$

I_{MR_2}' is the current in the rotor corresponding to I_{M_2}, which is due to the reverse field.

r_2' is the rotor resistance, *referred to the main winding.*

x_m is the magnetizing reactance.

x_2' is the rotor reactance, *referred to the main winding.*

The torques due to the two fields are opposing, so we have the torques, given by the usual expressions, viz.

$$T_1 = \frac{2 \times 112{\cdot}7}{\text{r.p.m.}_{\text{syn}}} \times (I_{m_2}')^2 \times \frac{r_2'}{s} \text{ oz-ft} \quad . \quad (5.102)$$

and $$T_2 = 2 \times \frac{112{\cdot}7}{\text{r.p.m.}_{\text{syn}}} \times (I_{MR_2}')^2 \times \frac{r_2'}{2-s} \text{ oz-ft} \quad . \quad (5.103)$$

and the total torque $= T_1 - T_2$

The starting torque is obtained by putting $s = 1$ in the above expression.

The starting torque will have a maximum value when the reactance of the capacitor equals $\frac{1}{\alpha^2}$ times the starting impedance of the main winding.

The maximum starting torque

$$= \frac{p}{\omega} \frac{V^2 r_2'}{R^2 + X^2} \times \alpha \times \frac{X}{R} \quad . \quad . \quad . \quad (5.104)$$

where $R + jX$ is the locked impedance of the motor referred to the main winding. The maximum torque varies as α, which is the ratio of effective turns in the main winding to those of the starting winding.*

There is still another method of running a motor from a single-phase supply, which is extensively used on very small motors, such as servo motors and those used for gyro work. In effect this method

* See *Application of the Method of Symmetrical Components*, Waldo V. Lyon. McGraw-Hill (New York: 1927)

uses a three-phase winding on the stator, connected in the following manner. It is known as the capacitor split method.

The three phases of the motor are shown as A, B, and C in Fig. 5.8, and the capacitor C is connected between P and Q.

It is possible to get, from the single-phase supply, a three-phase system of voltages, but the system is only symmetrical provided certain conditions are met. Let the three phases A, B, and C have

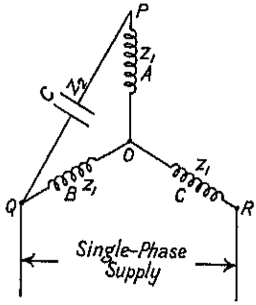


FIG. 5.8

total impedances per phase $\mathbf{Z}_1$ ohms and let $\mathbf{Z}_2$ = impedance of the capacitance. It will be assumed to have no loss, so $\mathbf{Z}_c = -jX_c$, where $X_c = \dfrac{1}{C\omega}$

The combined impedance, between Q and O

$$= \frac{\mathbf{Z}_1(\mathbf{Z}_1 + \mathbf{Z}_2)}{2\mathbf{Z}_1 + \mathbf{Z}_2} \qquad . \quad . \quad . \ (5.105)$$

and the impedance between Q and R

$$= \frac{\mathbf{Z}_1(\mathbf{Z}_1 + \mathbf{Z}_2)}{2\mathbf{Z}_1 + \mathbf{Z}_2} + \mathbf{Z}_1 \qquad . \quad . \quad . \ (5.106)$$

$$= \frac{3\mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_1 + \mathbf{Z}_2} \qquad . \quad . \quad . \ (5.107)$$

Let $\bar{I}_T$ be the current in the line when a sinusoidal voltage $\bar{E} \sin \omega t$ is applied between Q and R.

Then
$$\bar{I}_T = \frac{\bar{E} \times (2\mathbf{Z}_1 + \mathbf{Z}_2)}{3\mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2} \qquad . \quad . \quad . \ (5.108)$$

$$\text{The current in the leg } OQ = \bar{I}_T \times \frac{\mathbf{Z}_1 + \mathbf{Z}_2}{2\mathbf{Z}_1 + \mathbf{Z}_2} \qquad . \ (5.109)$$

$$\text{The current in the leg } OR = \bar{I}_T \qquad . \quad . \quad . \ (5.110)$$

$$\text{The current in the leg } OP = \bar{I}_T \times \frac{\mathbf{Z}_1}{2\mathbf{Z}_1 + \mathbf{Z}_2} \qquad . \ (5.111)$$

If the three currents in the three phases are to be equal, then

$$\left|\frac{\mathbf{Z}_1}{2\mathbf{Z}_1+\mathbf{Z}_2}\right| = \left|\frac{\mathbf{Z}_1+\mathbf{Z}_2}{2\mathbf{Z}_1+\mathbf{Z}_2}\right| = 1 \quad . \quad . \quad (5.112)$$

$$\therefore \qquad |\mathbf{Z}_1+\mathbf{Z}_2| = |2\mathbf{Z}_1+\mathbf{Z}_2| \quad . \quad . \quad . \quad (5.113)$$

i.e.

$$|R+jX-jX_c| = |2R+2jX-jX_c| \quad . \quad (5.114)$$

or

$$R^2+X^2+X_c^2-2XX_c = 4R^2+4X^2-4XX_c+X_c^2 \quad (5.115)$$

i.e.

$$X_c = \frac{3(R^2+X^2)}{2X} \quad . \quad . \quad . \quad (5.116)$$

The currents will be equal in magnitude, provided

$$X_c = \frac{3(R^2+X^2)}{2X} \quad . \quad . \quad . \quad (5.117)$$

They must also differ in phase by 120°.

In the branch OPQ, $\tan\phi_1 = \dfrac{X-X_c}{R}$. (5.118)

In the branch OQ, $\tan\phi_2 = \dfrac{X}{R}$

$$\tan(\phi_2-\phi_1) = \frac{\tan\phi_2-\tan\phi_1}{1+\tan\phi_2\tan\phi_1} = \frac{\frac{X}{R}-\left(\frac{X-X_c}{R}\right)}{1+\frac{X^2}{R^2}-\frac{XX_c}{R^2}} \quad (5.119)$$

$$= \frac{\frac{X_c}{R}}{1+\frac{X^2}{R^2}-\frac{XX_c}{R^2}} \quad . \quad . \quad . \quad . \quad (5.120)$$

Now if $\phi_2-\phi_1 = 120°$,

$$\tan 120 = -\sqrt{3}$$

$$\therefore \qquad \frac{X_c}{R} = -\sqrt{3}\left(1+\frac{X^2}{R^2}-\frac{XX_c}{R^2}\right) \quad . \quad . \quad (5.121)$$

$$\therefore \qquad X_c = \frac{\sqrt{3}\left(R+\frac{X^2}{R}\right)}{\sqrt{3}\frac{X}{R}-1} \quad . \quad . \quad . \quad (5.122)$$

The two values of X_0, given by equations (5.122) and (5.116), must be equal

i.e.
$$\frac{\sqrt{3}\left(R + \frac{X^2}{R}\right)}{\sqrt{3}\frac{X}{R} - 1} = \frac{3}{2}\left(\frac{R^2}{X} + X\right) \quad . \quad . \quad (5.123)$$

Let $\frac{X}{R} = K$, then from above $K = \sqrt{3}$

i.e.
$$\frac{X}{R} = \sqrt{3} \text{ and } X_0 = 2X \quad . \quad . \quad . \quad (5.124)$$

There are thus two conditions to be satisfied for a true three-phase system of voltages to be produced—

(a) $$\frac{X}{R} = \sqrt{3}$$

and (b) $$X_0 = 2X$$

The X is the *total* inductive reactance of each phase. R is the resistance of each phase. The capacitive reactance must be twice the total inductive reactance of each phase.

[*Note.* Total inductive reactance, *not leakage reactance.*]

Assuming these conditions are satisfied for a symmetrical three-phase system of voltages, we may enquire as to what the power factor may be. We have

$$I_T = E\frac{(2Z_1 + Z_2)}{3Z_1^2 + 2Z_1Z_2}$$

$$Z_1 = R + jX \quad . \quad . \quad . \quad (5.125)$$

$$Z_2 = -jX_0 = -2jX \quad . \quad . \quad . \quad (5.126)$$

Then
$$\frac{2Z_1 + Z_2}{3Z_1^2 + 2Z_1Z_2} = \frac{2R}{3R^2 + 2jRX + X^2} \quad . \quad . \quad (5.127)$$

and
$$I_T = \frac{\mathbf{E}2R[(3R^2 + X^2) - 2jRX]}{(3R^2 + X^2)^2 + 4R^2X^2} \quad . \quad . \quad (5.128)$$

It is assumed than $E = \hat{E}\sin\omega t$

and
$$\tan\alpha = \frac{2RX}{3R^2 + X^2} \quad . \quad . \quad . \quad (5.129)$$

When $\frac{X}{R} = \sqrt{3}$,

$$\tan\alpha = \frac{2\frac{X^2}{\sqrt{3}}}{3\frac{X^2}{3} + X^2} = \frac{\frac{2}{\sqrt{3}}X^2}{2X^2} = \frac{1}{\sqrt{3}} \quad . \quad (5.130)$$

$\therefore$ $$\alpha = 30^\circ$$

and
$$\cos\alpha = 0{\cdot}866 \quad . \quad . \quad . \quad (5.131)$$

Thus, the power factor is 0·866 for the line current.

It is clear that, under these conditions, the performance can be estimated in the same manner as that used for a three-phase motor under a balanced system of voltages. This method of starting and running is common in small induction motors for gyro work and also for small servo motors. It is also used for hysteresis motors. The condition for a balanced system of voltages is not met in practice, and one finds both inequality of voltages and phase angles differing

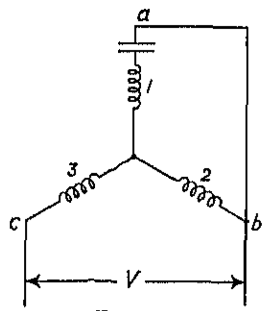


FIG. 5·9

from 120°. For the analysis of the performance, one can resort to the system of symmetrical components.*

As before, we obtain the positive- and negative-system of voltages and currents.

Let the capacitor reactance be $-jx$. Then the zero, positive- and negative-sequence components of this reactance $= -j\frac{x}{3}$

The applied potentials—

$$V_{ab} = 0; \quad V_{bc} = V; \quad V_{ca} = -V$$

$$V_{ab_1} = \tfrac{1}{3}[0 + aV - a^2V] \qquad (5\cdot132)$$

$$= \frac{V}{3}[a - a^2] = \frac{V}{3}\sqrt{3}j = j\frac{V}{\sqrt{3}}$$

$$a = -0\cdot5 + j0\cdot866 \qquad (5\cdot133)$$

$$a^2 = -0\cdot5 - j0\cdot866$$

$$V_{ab_2} = \tfrac{1}{3}[0 + a^2V - aV] \qquad (5\cdot134)$$

$$= \frac{V}{3}(-j)\sqrt{3} = -j\frac{V}{\sqrt{3}} \qquad (5\cdot135)$$

The positive-sequence component of the star voltage

$$V_{a_1} = \frac{V_{ab_1}}{\sqrt{3}}\angle -30 = \frac{V}{3}\angle 60 \qquad (5\cdot136)$$

$$V_{a_2} = \frac{V_{ab_2}}{\sqrt{3}}\angle 30 = \frac{V}{3}\angle -60 \qquad (5\cdot137)$$

* Op. Cit.

The positive-sequence voltage in phase 1, i.e. the phase containing the capacitor,

$$= V_{a_1} = \frac{V}{3} \angle 60$$

The negative-sequence voltage in phase 1

$$= V_{a_2} = \frac{V}{3} \angle -60$$

$\frac{V}{3} \angle 60$ means that the magnitude $= \frac{V}{3}$ and leads V by 60°

$\frac{V}{3} \angle -60$ means that the magnitude $= \frac{V}{3}$ and lags V by 60°

Now for the voltage equations for the positive and negative sequences for phase 1. We have

$$\mathbf{V}_{a_1} = \mathbf{I}_{a_1}\left(\mathbf{Z}_1 - j\frac{x}{3}\right) - j\frac{x}{3} I_{a_2} \qquad (5.138)$$

$$\mathbf{V}_{a_2} = \mathbf{I}_{a_2}\left(\mathbf{Z}_2 - j\frac{x}{3}\right) - j\frac{x}{3} I_{a_1} \qquad (5.139)$$

The impedance of phase 1 to the positive-sequence currents is $\mathbf{Z}_1$, and to negative-sequence currents is $\mathbf{Z}_2$.

Substituting the values of $\mathbf{V}_{a_1}$ and $\mathbf{V}_{a_2}$ already found, we have

$$\frac{V}{3} \angle 60 = \mathbf{I}_{a_1}\left(\mathbf{Z}_1 - j\frac{x}{3}\right) - j\frac{x}{3}\mathbf{I}_{a_2} \qquad (5.140)$$

$$\frac{V}{3} \angle -60 = \mathbf{I}_{a_2}\left(\mathbf{Z}_2 - j\frac{x}{3}\right) - j\frac{x}{3}\mathbf{I}_{a_1} \qquad (5.141)$$

or

$$\frac{V}{3} \angle -60 = \mathbf{I}_{a_1}\left(-j\frac{x}{3}\right) + \mathbf{I}_{a_2}\left(\mathbf{Z}_2 - j\frac{x}{3}\right) \qquad (5.142)$$

$$\therefore \qquad \mathbf{I}_{a_1} = \frac{\begin{vmatrix} \frac{V}{3}\angle 60 & -j\frac{x}{3} \\ \frac{V}{3}\angle -60 & \mathbf{Z}_2 - j\frac{x}{3} \end{vmatrix}}{\begin{vmatrix} Z_1 - j\frac{x}{3} & -j\frac{x}{3} \\ -j\frac{x}{3} & \mathbf{Z}_2 - j\frac{x}{3} \end{vmatrix}} \qquad (5.143)$$

$$\mathbf{I}_{a_1} = \frac{\frac{V}{3}\mathbf{Z}_2 \angle 60 - \frac{V}{3}\angle 60\, j\frac{x}{3} + \frac{V}{3} j\frac{x}{3} \angle -60}{Z_1 Z_2 - j\frac{x}{3}(Z_1 + Z_2)} \qquad (5.144)$$

$$\therefore \qquad I_{a1} = \frac{\frac{V}{3}\left[\mathbf{Z}_2 \angle 60 + \frac{x}{\sqrt{3}}\right]}{\mathbf{Z}_1\mathbf{Z}_2 - j\frac{x}{3}(\mathbf{Z}_1 + \mathbf{Z}_2)} \qquad . \qquad . \quad (5.145)$$

Note $\qquad \frac{V}{3}j \angle -60 \frac{x}{3} = \frac{V}{3} \angle 30 \frac{x}{3}$

and $\qquad -\frac{V}{3} \angle 60\, j\frac{x}{3} = -\frac{V}{3} \angle 150 \frac{x}{3}$

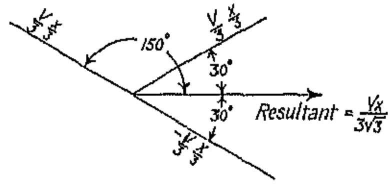


Fig. 5.10

$$I_{a_2} = \frac{\frac{V}{3}\left(\mathbf{Z}_1 \angle -60 - \frac{x}{\sqrt{3}}\right)}{\mathbf{Z}_1\mathbf{Z}_2 - j\frac{x}{3}[\mathbf{Z}_1 + \mathbf{Z}_2]} \qquad . \qquad . \quad (5.146)$$

The current in phase 3 is the line current

$$I_c = I_{c_1} + I_{c_2} \qquad . \qquad . \qquad . \qquad . \quad (5.147)$$

$$= aI_{a_1} + a^2 I_{a_2} \qquad . \qquad . \qquad (5.148)$$

a is an operator $= e^{j120} = \cos\ 120 + j\ \sin\ 120$, it turns the vector on which it operates through 120°.

$$\therefore \qquad I_c = I_{a_1} \angle 120 + I_{a_2} \angle 240$$

$$= I_{a_1} \angle 120 + I_{a_2} \angle -120$$

$$\therefore \qquad I_c = \frac{-V[\mathbf{Z}_1 + \mathbf{Z}_2 - jx]}{3[\mathbf{Z}_1\mathbf{Z}_2] - jx[(\mathbf{Z}_1 + \mathbf{Z}_2)]} \qquad . \qquad . \quad (5.149)$$

To make this step clear, we have

$$I_c = \frac{\frac{V}{3}\left[\mathbf{Z}_2 \angle 180 + \frac{x}{\sqrt{3}} \angle 120\right]}{\mathbf{Z}_1\mathbf{Z}_2 - j\frac{x}{3}(\mathbf{Z}_1 + \mathbf{Z}_2)} + \frac{\frac{V}{3}\left[\mathbf{Z}_1 \angle -180 - \frac{x}{\sqrt{3}} \angle -120\right]}{\mathbf{Z}_1\mathbf{Z}_2 - j\frac{x}{3}(\mathbf{Z}_1 + \mathbf{Z}_2)} \qquad . \qquad . \quad (5.150)$$

$$= V\left[\frac{\mathbf{Z}_2 \angle 180 + \mathbf{Z}_1 \angle -180 + \frac{x}{\sqrt{3}} \angle 120 - \frac{x}{\sqrt{3}} \angle -120}{3\mathbf{Z}_1\mathbf{Z}_2 - jx(\mathbf{Z}_1 + \mathbf{Z}_2)}\right] \qquad . \qquad . \quad (5.151)$$

$$= \frac{V[-Z_2 - Z_1 + jx]}{3\mathbf{Z}_1\mathbf{Z}_2 - jx(\mathbf{Z}_1 + Z_2)} = -\frac{V[\mathbf{Z}_2 + \mathbf{Z}_1 - jx]}{3\mathbf{Z}_1\mathbf{Z}_2 - jx(\mathbf{Z}_1 + \mathbf{Z}_2)} \quad . \ (5.152)$$

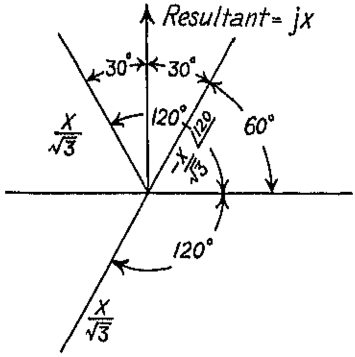


Fig. 5.11

Note. $\frac{x}{\sqrt{3}} \angle 120 - \frac{x}{\sqrt{3}} \angle - 120$ has a resultant $\frac{2x}{\sqrt{3}} \cos 30j$ as shown in Fig. 5.11

$$= \frac{2x}{\sqrt{3}} \frac{\sqrt{3}j}{2} = xj$$

Our positive-sequence diagram then becomes as in Fig. 5.12.

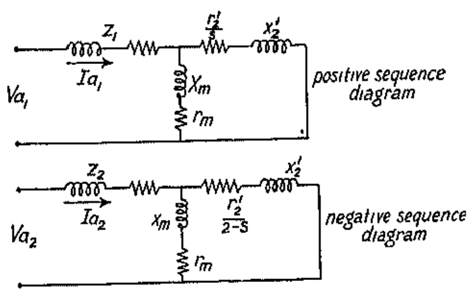


Fig. 5.12

The positive-sequence current in the rotor

$$= -I_{a_1} \times \frac{Z_m}{Z_m + \frac{r_2'}{s} + jx_2'} \simeq -I_{a_1} \times \frac{jx_m}{\frac{r_2'}{s} + j(x_m + x_2')} = I_{a_{1R}}' \quad . \quad . \ (5.153)$$

The negative-sequence current in the rotor

$$= -I_{a_1} \times \frac{jx_m}{\frac{r_2'}{2-s} + j(x_2' + x_m)} = + I_{a_{2R}}' \quad . \ (5.154)$$

where r_2' = rotor resistance, referred to the stator

x_2' = rotor reactance, at full frequency, referred to the stator

m_2 = number of rotor phases

The torque of the motor, in synchronous watts,

$$= m_2\left[(I_{a_{1R}}')^2\frac{r_2'}{s} - (I_{a_{2R}}')^2\frac{r_2'}{2-s}\right] \quad . \quad . \quad (5.155)$$

The starting torque, in synchronous watts,

$$= m_2[(I_{a_{1R}}')^2 r_2' - (I_{a_{2R}}')^2 r_2'] \quad . \quad . \quad (5.156)$$

$$\simeq m_2[(I_{a_1}')^2 r_2' - (I_{a_2}')^2 r_2'] \quad . \quad . \quad . \quad (5.157)$$

$$\simeq m_2\frac{V^2}{9}\left[\frac{\left(\mathbf{Z}\angle 60 + \frac{x}{\sqrt{3}}\right)^2 - \left(\mathbf{Z}\angle -60 - \frac{x}{\sqrt{3}}\right)^2}{\left(\mathbf{Z}^2 - j\frac{x}{3}2\mathbf{Z}\right)^2}\right]r_2' \quad (5.158)$$

Since, when $s = 1$, $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}$.

The starting torque then becomes

$$\simeq m_2\frac{V^2}{9}\frac{\frac{2}{\sqrt{3}}Rxr_2'}{(R^2+X^2)\left(R^2+\left\{X-\frac{2x}{3}\right\}^2\right)} \quad . \quad (5.159)$$

where R = resistance per phase

and X = the reactance per phase at standstill

Differentiating equation (5.159) with respect to x, we have

$$\frac{R^2+\left(X-\frac{2x}{3}\right)^2+2\left(X-\frac{2x}{3}\right)\times\frac{2}{3}x}{\left\{R^2+\left(X-\frac{2x}{3}\right)^2\right\}^2}$$

for a maximum value

$$R^2+X^2-\frac{4xX}{3}+\frac{4x^2}{9}+\frac{4Xx}{3}-\frac{8}{9}x^2 = 0 \quad . \quad (5.160)$$

i.e. $$R^2+X^2-\frac{4}{9}x^2 = 0$$

i.e. $$x^2 = \frac{9}{4}(R^2+X^2) \quad . \quad . \quad . \quad (5.161)$$

and $$x = 1\cdot 5\sqrt{R^2+X^2}$$

That is, the starting torque is a maximum when the reactance of the capacitor $= 1\cdot5 \times$ impedance per phase.

Substituting we have, maximum torque in synchronous watts

$$= \frac{m_2 V^2}{6\sqrt{3}} \frac{R r_2'}{(R^2 + X^2)(\sqrt{R^2 + X^2} - X)} \quad . \quad (5\cdot162)$$

The maximum torque at the start in lb-ft

$$= \frac{7\cdot04}{\underset{\text{syn}}{\text{r.p.m.}}} \times \frac{m_2 V^2 R r_2'}{(R^2 + X^2)(\sqrt{R^2 \times X^2} - X) \times 6\sqrt{3}} . \quad (5\cdot163)$$

If $X \gg R$

$$\sqrt{X^2 + R^2} - X = X\sqrt{1 + \frac{R^2}{X^2}} - X$$

$$\simeq X\left(1 + \frac{1}{2}\frac{R^2}{X^2}\right) - X \simeq \frac{R^2}{2X}$$

and the maximum starting torque

$$= \frac{7\cdot04}{\underset{\text{syn}}{\text{r.p.m.}}} \times \frac{m_2}{6\sqrt{3}} \frac{V^2 r_2' \times 2X}{(R^2 + X^2)R} \quad . \quad . \quad . \quad (5\cdot164)$$

$$= \frac{7\cdot04}{\underset{\text{syn}}{\text{r.p.m.}}} \times \left(\frac{V}{\sqrt{3}}\right)^2 \times \frac{m_2 r_2'}{R^2 + X^2} \times \frac{X}{\sqrt{3}R} \quad . \quad (5\cdot165)$$

Thus, the maximum torque developed at the start $= \dfrac{X}{\sqrt{3}R}$ times that for the normally balanced motor.

CHAPTER VI

Electric Braking of Induction Motors

In certain classes of machines, such as hoists, cranes, rolling mills, etc., it is necessary to stop quickly, and hence braking is necessary. An induction motor may be braked in several ways. Alternating current may be used for excitation, but it is necessary to reverse the field of the motor by reversing two of the primary leads in the three-phase machine, and by reversing two primary leads of one phase in a two-phase machine. The rotor is then running in the opposite direction to the field. The power generated in the rotor is dissipated in I^2R loss. Remembering that the torque, in synchronous watts, is equal to the rotor input, i.e. the torque, in synchronous watts

$$= (I_{2t}')^2 \frac{r_2'}{s} \qquad . \quad . \quad . \quad . \quad (6.1)$$

Note. I_{2t}' in this case $= \sqrt{3} \times$ rotor current per phase referred to the stator

and
$$\text{torque in lb-ft} = \frac{(I_{2t}')^2 r_2'}{\text{r.p.m.}_{\text{syn}} \times s} \times 7{\cdot}04 \qquad . \quad . \quad (6.2)$$

we see that, if the torque is to remain constant, during the braking period, and I_{2t}' is constant, then $\frac{r_2'}{s}$ must be constant. As the slip decreases from $s = 2$, which corresponds to full speed, with the field rotating in the opposite direction, to $s = 1$, i.e. to standstill, if the rotor current is to remain constant, with constant braking torque, *then r_2' must decrease in the same ratio as the slip decreases.* If a liquid resistance is used, it would be possible to change the resistance in such a manner that the torque and current would remain constant during the braking period. At the instant when the motor is running near synchronous speed, and the field is reversed, the frequency of the rotor currents is nearly twice that at standstill, and the secondary voltage will be nearly twice that at standstill. In large induction motors, this may impose excessive stress on the insulation, so in these machines it is desirable to apply half normal voltage to the primary, or use direct current for braking. We will now develop

approximate expressions for the speed-torque and ampere-torque curves.

We will use our approximate equivalent circuit, shown in Fig. 6.1.

Let $\bar{V}_1$ = primary terminal volts; the machine is assumed to be connected in star, so that the phase voltage $= \dfrac{\bar{V}_1}{\sqrt{3}}$

$\bar{V}_2$ = secondary volts between rings at standstill

$\bar{I}_1 = \sqrt{3} \times$ current per phase, for three-phase

$= 2 \times$ current per line, for two-phase

$\bar{I}_{0t} = \sqrt{3} \times$ no-load current

$\bar{I}_{2t} = \sqrt{3} \times$ secondary current per phase

Note. Since we are using the terminal volts in our equivalent circuit, the current to be used in calculating power $= \sqrt{3} \times$ current per phase.

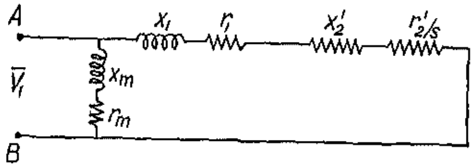


FIG. 6.1

r_1 = primary resistance per phase

r_2' = secondary resistance, referred to the primary

x_1 = primary leakage reactance per phase

x_2' = the secondary leakage reactance per phase, referred to the primary

ρ = total watts with the motor locked

τ = torque in lb-ft

s = slip

n_0 = synchronous revolutions per minute

Since the power factor at no load and at short circuit is nearly the same,

$$x_1 + x_2' \doteq \frac{\sqrt{\{(\bar{I}_{st} - \bar{I}_{0t})\bar{V}_1\}^2 - \rho^2}}{(\bar{I}_{st} - \bar{I}_{0t})^2} \quad . \quad (6.3)$$

where $\bar{I}_{st} = \sqrt{3} \times$ short-circuit current per phase

$$\bar{I}_{2t}' = \frac{\bar{V}_1}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}} \quad . \quad . \quad . \quad (6.4)$$

$$\tau = \text{torque in lb-ft} = \frac{(I_{2t}')^2 \times \frac{r_2'}{s} \times 7{\cdot}04}{n_0} \quad . \qquad . \quad (6.5)$$

$$\therefore \qquad \text{slip } s = \frac{(I_{2t}')^2 r_2' \times 7{\cdot}04}{n_0 \times \tau} \quad . \quad . \quad . \quad (6.6)$$

By substituting for I_{2t}' the value

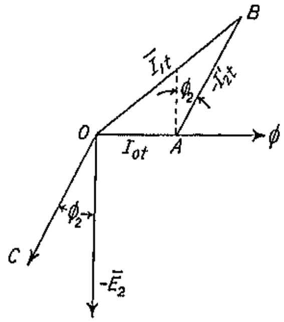


FIG. 6.2

$$\frac{V_1}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}}$$

in equation (6.6), and differentiating, we find the torque is a maximum when

$$r_1 + \frac{r_2'}{s} = x_1 + x_2'$$

Neglecting the no-load watt-component of the primary current, our vector diagram, Fig. 6.2, shows the primary current $(\times \sqrt{3}) = OB$,

and $$I_{1t} = \sqrt{(I_{2t}' \sin \phi_2 + I_{0t})^2 + (I_{2t}' \cos \phi_2)^2} \quad . \quad (6.7)$$

and since $$\sin \phi_2 = \frac{x_1 + x_2'}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}} \quad . \quad . \quad (6.8)$$

and $$\cos \phi_2 = \frac{r_1 + \frac{r_2'}{s}}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}} \quad . \quad . \quad (6.9)$$

we have

$$I_{1t} = \sqrt{\left\{\frac{I_{2t}'(x_1 + x_2')}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}} + I_{0t}\right\}^2 + \left\{\frac{I_{2t}'\left(r_1 + \frac{r_2'}{s}\right)}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}}\right\}^2} \quad . \quad . \quad (6.10)$$

From these equations the torque-slip and torque-current curves can be drawn for various values of the rotor resistance.

Note. The torque in synchronous watts = input to the rotor circuit

$$= \frac{m_2(I_2')^2 r_2'}{s} \quad . \quad . \quad . \quad . \quad (6.11)$$

For a three-phase rotor $m_2 = 3$, but

$$I_{2t}' = \sqrt{3}I_2' \quad . \quad . \quad . \quad . \quad (6.12)$$

$$\therefore \qquad \frac{m_2(I_2')^2 r_2'}{s} = \frac{(I_{2t}')^2 \times r_2'}{s} \quad . \quad . \quad . \quad (6.13)$$

The following example is given by Specht, to whose article on braking, the reader is referred (*Journal of The American Institute of Electrical Engineers*).

A motor of 2000 b.h.p., three-phase, 6600 V line, 25 c/s, 6 poles, 500 r.p.m. (syn.), with both stator and rotor connected in star, is braked with a.c. current.

$$I_{0t} = 58 \text{ A} = \text{line amperes} \times \sqrt{3}$$

No-load power = 32 kW

$$I_{st} = 1550 \text{ A with locked rotor} = \sqrt{3} \times I_s$$

$$\rho = 1950 \text{ kW with locked rotor}$$

$$r_1 = 0{\cdot}38\ \Omega$$

$$E_2 = 1700 \text{ V between rings}$$

The resistance per phase of the secondary at 40° = 0·026 Ω,

$$\therefore \qquad r_2' = 0{\cdot}026 \times \left(\frac{6600}{1700}\right)^2 = 0{\cdot}39\ \Omega \quad . \quad . \quad (6.14)$$

In Fig. 6.3 nine speed-torque curves are given for the following rotor resistances, in ohms, referred to the primary—

Curve 1. $r_2' = 0{\cdot}39$ without external resistance
,, 2. $r_2' = 1{\cdot}2$ including external resistance
,, 3. $r_2' = 2{\cdot}4$,, ,, ,,
,, 4. $r_2' = 5{\cdot}0$,, ,, ,,
,, 5. $r_2' = 8{\cdot}0$,, ,, ,,
,, 6. $r_2' = 13{\cdot}0$,, ,, ,,
,, 7. $r_2' = 20{\cdot}0$,, ,, ,,
,, 8. $r_2' = 34{\cdot}0$,, ,, ,,
,, 9. $r_2' = 60{\cdot}0$,, ,, ,,

For $r_2' = 0{\cdot}39$ and $s = 2$, we have

$$I_{2t} = \frac{6600}{\sqrt{\left(0{\cdot}38 + \frac{0{\cdot}39}{2}\right)^2 + 4{\cdot}37^2}} = 1496 \text{ A} \qquad (6.15)$$

$$x_1 + x_2 = \frac{\sqrt{[(1550 - 58) \times 6600]^2 - (1\,950\,000)^2}}{(1550 - 58)^2} = 4{\cdot}37\ \Omega \quad (6.16)$$

and

$$\tau = \frac{1496^2 \times \frac{0{\cdot}39}{2}}{500} \times 7{\cdot}04 = 6100 \text{ lb-ft} \quad . \quad . \quad (6.17)$$

For various values of the slip from 2 to 0, the torque curve 1 and secondary ampere curve are determined.

The results are shown in Fig. 6.3.

It will be noticed that if the motor should be braked with a torque equal to the average full-load torque, and with a current not exceeding greatly the full-load current, the resistance should be decreased, step by step, until the motor stops. This is shown by the heavy zigzag line. This corresponds to the use of a metallic resistance. If a reduced voltage is applied to the stator for braking, the currents vary in the same ratio as the voltage, and the torque varies as the square of the voltage ratio. Therefore, the same curves can be used for reduced voltage as for full voltage, by changing the current scale in the ratio of the voltage and the torque scale in the square of the voltage ratio. In order to brake the motor at half-voltage with full-load current and half the full-load torque, the secondary resistance must be half that at full voltage. This torque can only be obtained when the maximum torque is at least twice full-load torque at full voltage, because the maximum torque at half-voltage will be only one-quarter of that at full voltage.

In the case of a squirrel-cage motor, one speed-torque curve only can be obtained, and in order to obtain a good torque for braking, without excessive current, a high-resistance rotor is necessary. Such a motor is very inefficient when running normally, and should only be used for very intermittent service for elevators, cranes, and hoists.

Braking by Direct Current

When using direct current for braking, the d.c. supply may be connected to the primary in the following ways, shown in Fig. 6.4. The connections for three-phase machines are given in Figs. (*a*) to (*f*), and those for two-phase machines in (*g*) to (*h*).

In the case where one terminal is connected to the negative supply and the remaining two terminals to the positive supply, the current in phase 1 (when at the maximum value) will have double the value of the currents in phases 2 and 3, and will flow in opposite direction. The current in phases 2 and 3 flows from the terminals to the star point (negative direction) and, in phase 1, it will flow from the star point to the terminal (positive direction).

This is just what happens when, in a three-phase system, the current in phase 1 is at its crest value. Let us call the r.m.s. value of the current I, then the momentary current in phase 1 is $\sqrt{2}I$ and, in phases 2 and 3, $\frac{1}{\sqrt{2}} I$

It is clear that the direct current, in this case, which replaces the three-phase excitation, will have the value $\sqrt{2}I$

Let us consider a star-wound motor and connect only two

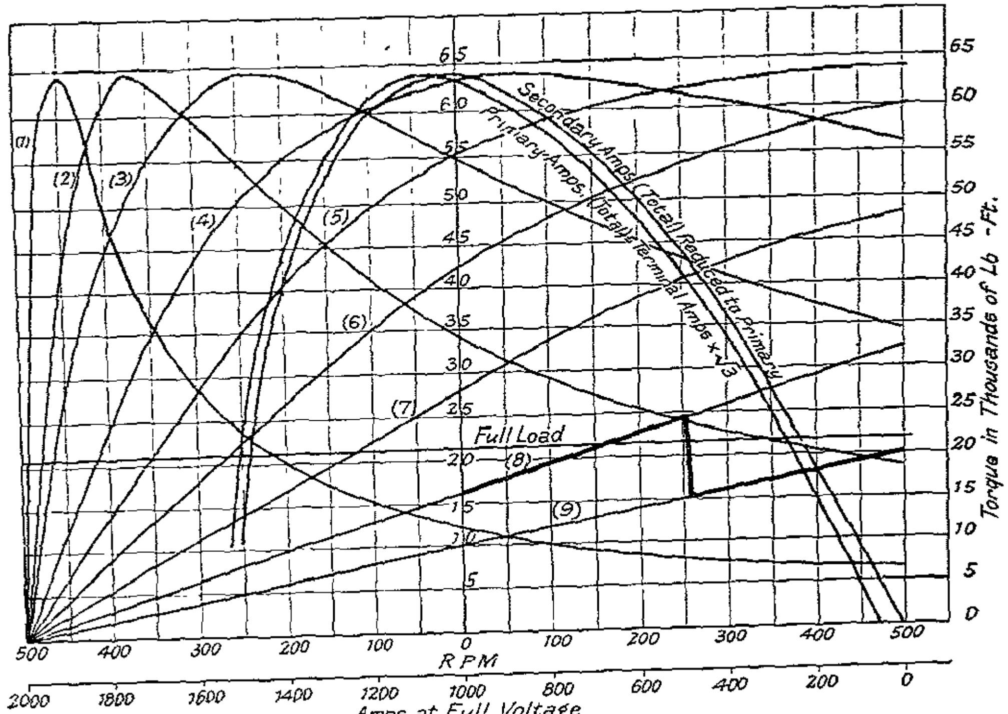


Fig. 6 3. Ampere Speed/Torque Curve for A.C. Excitation

terminals to the supply. Then the current in the third phase is zero, the currents in phases 1 and 2 have the same value, but opposite in direction. This corresponds to the moment in which, in the three-phase machine, the current in phase 3 is zero, whereas the currents in the other two phases are

$$+\sqrt{\frac{3}{2}}\,I \text{ and } -\sqrt{\frac{3}{2}}\,I = 1{\cdot}23I \text{ and } -1{\cdot}23I$$

In this case the equivalent direct current will be 23 per cent higher than the measured value of the alternating current.

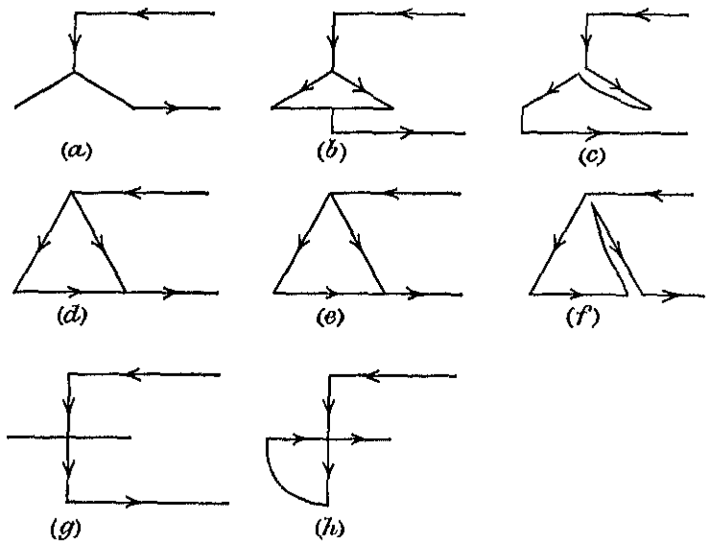


FIG. 6.4

The watt losses in the first case are, if R represents the resistance of one phase,

$$(1{\cdot}414I)^2R + 2 \times (0{\cdot}707I)^2R = 3I^2R \quad . \quad . \quad (6.18)$$

and, in the second case,

$$2(1{\cdot}23I)^2R = 3I^2R \quad . \quad . \quad . \quad (6.19)$$

Therefore, the d.c. voltages, applied in cases 1 and 2, are in inverse proportion to the currents.

Exactly the same relation holds good for delta connection. In this case, if the d.c. supply is connected to two terminals only, the direct current has to be 23 per cent greater than the measured alternating current; whereas if one terminal is connected to the positive and the two others to the negative supply, the value of the current is $\sqrt{2}I$

Let r_2 = secondary ohmic resistance measured between terminals and divided by 2

x = inductive reactance at synchronous speed

s = slip = 0 at synchronous speed of motor
= 1 at standstill

n_0 = synchronous revolutions per minute

n = revolutions per minute of motor when running

e_2 = secondary voltage between terminals at no-load speed

i_2 = *total* secondary current equivalent to single phase
[for three-phase i_2 = terminal amperes $\times \sqrt{3}$]
[for two-phase i_2 = terminal amperes $\times$ 2]

i_s = total secondary short-circuit current for inductive reactance purely

i_1 = total primary amperes

T = torque in lb at 1 ft radius

i_0 = direct current for exciting

t_1 = number of turns per phase in primary

t_2 = number of turns per phase in secondary

For the d.c. excitation, an equivalent alternating current $= \frac{1}{\sqrt{2}}$ times the direct current can be substituted, and then, for synchronous speed, the voltage and short-circuit current in the secondary can be determined by transformation.

The short-circuit current is equal to the equivalent alternating magnetizing current reduced to the secondary turns. The secondary open-circuit voltage is the same as that obtained by the equivalent d.c. excitation when running at synchronous speed. In determining the corresponding alternating short-circuit current, the distribution and the amount of winding excited by direct current must be considered, i.e. the current must be multiplied by a factor C besides $\frac{1}{\sqrt{2}}$

The factor C for a three-phase winding, of which two phases are excited by direct current, is 1·15.

The short-circuit current in a three-phase secondary at synchronous speed, which would be obtained if the rotor had inductive reactance only is

$$i_s = \frac{i_0\sqrt{3}}{\sqrt{2}} \times \frac{t_1}{t_2} \times C \quad . \quad . \quad . \quad (6.20)$$

and the equivalent primary alternating current is

$$i_1 = i_0 \times \frac{\sqrt{3}}{\sqrt{2}} \times C \text{ for three-phase primary} \quad . \quad (6.21)$$

$$i_1 = i_0 \frac{2}{\sqrt{2}} \times C \text{ for two-phase primary} \quad . \quad (6.22)$$

The corresponding secondary volts e_2 can be read off on the open a.c. saturation curve.

The inductive reactance $$x = \frac{e_2}{i_s} \qquad (6.23)$$

also $$i_2 = \frac{e_2}{\sqrt{\left(\frac{r_2}{s}\right)^2 + x^2}} \qquad (6.24)$$

$$T = \frac{i_2{}^2 r_2 \times 7{\cdot}04}{n} \qquad (6.25)$$

$$s = \frac{n_0 T}{i_2{}^2 r_2} \times 7{\cdot}04 \qquad (6.26)$$

Maximum torque occurs when $\frac{r_2}{s} = x$.

We will compare d.c. braking with a.c. braking.

The 2000 h.p. motor, to which we have already referred, will be selected for example. Assuming the primary is excited by direct current across two terminals of the star winding, and that this current is 100 A, then

$$i_s = 100 \times \frac{\sqrt{3}}{\sqrt{2}} \times \frac{6600}{1700} \times 1{\cdot}15 = 550 \text{ A total} \qquad (6.27)$$

$$i_1 = 100 \times \frac{\sqrt{3}}{\sqrt{2}} \times 1{\cdot}15 = 141{\cdot}7 \text{ A total} \qquad (6.28)$$

From the saturation curve, the corresponding secondary voltage e_2 for $i_1 = 141{\cdot}7$ A is found to be 2420 V.

The inductive reactance $= \frac{2420}{550} = 4{\cdot}4\ \Omega$.

For various secondary resistances r_2, the following values are selected—

Curve 1. $r_2 = 0{\cdot}026\ \Omega$ without external resistance
,, 2. $r_2 = 0{\cdot}5\ \Omega$ including external resistance
,, 3. $r_2 = 1{\cdot}0\ \Omega$,, ,, ,,
,, 4. $r_2 = 2{\cdot}0\ \Omega$,, ,, ,,
,, 5. $r_2 = 3{\cdot}0\ \Omega$,, ,, ,,

For $s = 1$ and $r_2 = 0{\cdot}5\ \Omega$, the secondary current i_2 and torque T are

$$i_2 = \frac{2420}{\sqrt{\left(\frac{0{\cdot}5}{1{\cdot}0}\right)^2 + 4{\cdot}4^2}} = 548 \text{ A} \qquad (6.29)$$

$$T = \frac{548^2 \times 0{\cdot}5}{500} \times 7{\cdot}04 = 2120 \text{ lb-ft} \qquad (6.30)$$

In this way the currents and torques for other slips and resistances may be found. The results are shown in the following figure.

Examination of the curves shows that, without external resistance, the torque at 500 r.p.m. is nearly zero, and increases very slowly with decreasing speed, except that below 50 r.p.m. the torque increases much faster. At about 3 r.p.m. the torque reaches the maximum value (9400 lb-ft) and then drops very rapidly to zero value. It is clear that this speed-torque curve is of no practical

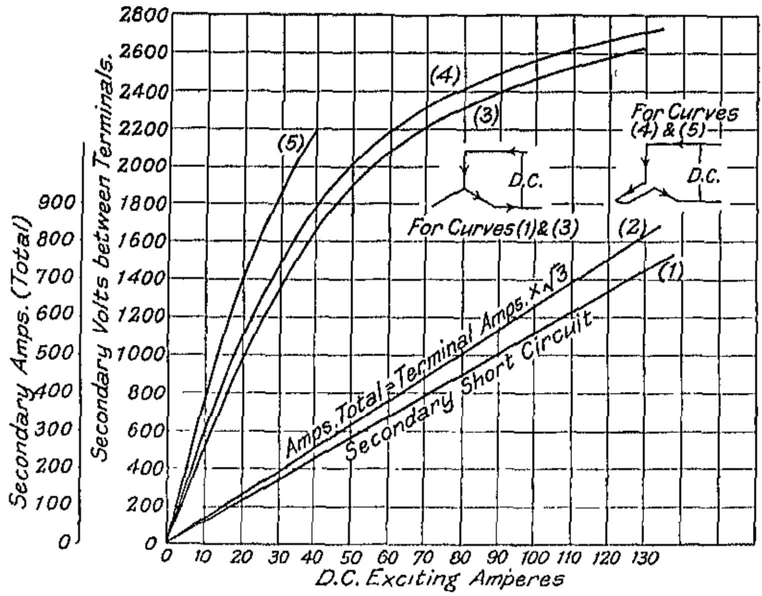


FIG. 6.5. SATURATION CURVE FOR A.C. EXCITATION

value; and in order to obtain good braking torque over a wide speed range, it is necessary to insert a fairly large resistance.

Further, the curves show that the maximum torque obtainable with an exciting current of 100 A is not even quite half full-load torque, and that the open-circuit secondary voltage at synchronous speed is 42 per cent greater than the voltage at standstill with 6600 V alternating current on the primary. It would be possible to obtain a greater torque by increasing the exciting current. This would give, however, a higher secondary voltage and stronger field, and would increase the unbalanced pull and the danger o greater potential rise in case any of the circuits should break. It is pointed out by Specht that the conditions for d.c. braking of this motor are very poor, due to the low value of the no-load current as compared to the full-load current. Motors having a larger ratio in this respect would give more favourable results.

Nevertheless, the best torque which can be obtained by braking

with direct current is not greater than full-load torque. The same motor was driven by another motor at synchronous speed, and the primary was excited by direct current and the open-circuit secondary voltage measured. Then the secondary circuit was closed and the short-circuit current was measured. The results are shown in Fig. 6.5.

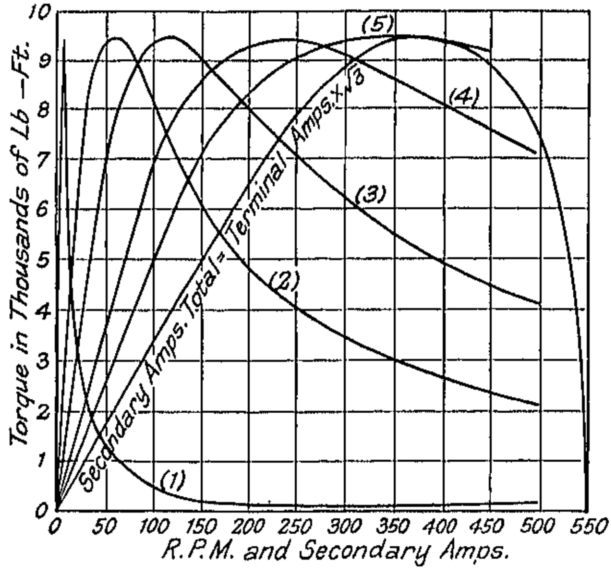


FIG. 6.6. AMPERE SPEED/TORQUE CURVE FOR D.C. EXCITATION

Curves 1 and 3 correspond to an excitation of two phases; and curves 2, 4, and 5 correspond to an excitation of three phases.

Analysis of the results shows—

(1) The braking torque obtainable by alternating current, even with only half the primary voltage, is, as a rule, considerably greater than with direct current.

(2) In braking with alternating current, the line circuit has to be taken off the motor as soon as the motor comes to rest, otherwise the motor will reverse; whereas in braking with direct current, the motor comes to rest only and will not reverse.

(3) With alternating current, it is an easy matter to obtain a strong and practically constant braking torque during the whole retardation period; whereas with d.c. excitation, it is difficult to obtain good braking torque near standstill, due to the rapid decrease in torque from maximum to zero.

(4) If it is desired to brake the motor with full-load torque by means of direct current, the secondary voltage at synchronous speed will be not far from double voltage; and, further, since the magnetic field has to be much stronger with d.c. current, there is a danger of serious voltage rises due to breaking of any of the circuits.

The only advantage of braking with direct current is the small energy which is needed. Only the I^2R losses of the primary have to be supplied with d.c. excitation; while with a.c. excitation, the full power has to be supplied to the motor which it would require for developing an equal torque at normal operating condition. This section with the curves and examples are taken from the paper by Specht.

CHAPTER VII

Speed Control of Induction Motors

The induction motor is practically a constant-speed motor, resembling, in this respect, the d.c. shunt motor, and is admirably adapted to constant-speed work, but there are many, and varied, applications where variable speed is necessary. This variable-speed field is extensive and includes rolling-mill motors, cranes and hoists, pumps and compressors, etc. There is a trend towards the use of alternating current on ships, and variable-speed motors are necessary for winches, capstans, etc. There are numerous methods by which speed control can be obtained with the induction motor. They are as follows—

(*a*) Rheostatic control by the introduction of resistance in the rotor circuits.

(*b*) Pole-changing.

(*c*) Cascade connection.

(*d*) Change of supply frequency.

(*e*) By the use of a synchronous converter in circuit with the rotor.

(*f*) By concatenation with the three-phase series and shunt commutator motors.

(*g*) By the use of resistance and reactance in parallel in the rotor circuits.

Speed Control by Resistance in Rotor Circuit

The simplest method of varying the speed of an induction motor is to insert resistance in the rotor circuit. By this means any speed required, *below* synchronous speed is obtainable, but only at the cost of efficiency. It has been shown that the percentage efficiency is always less than the speed as a percentage of synchronous speed. It is chiefly used in the following cases—

(1) During starting a motor, where the energy wasted in resistance bears a small ratio to the total energy.

(2) For the speed control of haulage motors of medium size, where the wind is a long one.

(3) For rolling-mill motors and winding-gear equalizers.

The Continuous Slip Regulator

For rolling-mill work, where a mill has to roll a great variety of sections, it is clear that speed regulation is necessary. Smaller sections must be finished at a higher speed than the larger ones, otherwise the metal would cool too rapidly and could only be formed by the expenditure of a great deal of power, which increases the liability of the breakdown of the mill, and the accuracy of the sections and quality of the product may be affected. To obtain a reasonable production, the smaller sections must be rolled at as high a speed as possible. Within certain limits, where the speed regulation does not exceed 10 to 15 per cent, the rheostatic method of control is the simplest and most satisfactory.

In rolling-mill applications, it is very important to equalize the load on the generating plant as much as possible, to ensure maximum efficiency in operation. To that end, a flywheel is coupled to the induction motor, which is caused to drop its speed by the introduction of resistance in the rotor circuits. This allows the flywheel to give up some of its energy when peak loads have to be met. Thus, during the passes, a sudden, heavy demand is made on the motor and generating system. This demand on the generating plant is reduced by the equalizing effect of the flywheel, and, furthermore, the size of motor is also reduced. The resistance is introduced into the rotor circuit in two ways. In one the resistance is *permanently* in circuit, and the method is known as the "continuous slip regulator." It is a matter of interest to analyse the behaviour of the motor, and to show how the speed, torque, and output of the motor, and input from the line vary under load conditions. We will consider, in the first place, a rolling-mill of the continuous type, and deduce the torque and speed equations when permanent resistance is adopted.

Let I = moment of inertia of the flywheel in lb-ft^2

ω = angular velocity in radians per second at any time t seconds from the commencement of the pass

T = torque exerted by the induction motor at any time t seconds, from the commencement of the pass, in lb-ft

T_f = full-load torque in lb-ft

T_2 = maximum torque exerted by the induction motor in the pass in lb-ft

T_3 = total torque in the pass, assumed constant, in lb-ft

ω_0 = synchronous speed in radians per second

ω_f = speed of the motor, in radians per second, at full-load, with permanent resistance in circuit

E_2 = voltage per phase in the rotor, at standstill

I_2 = rotor current per phase in amperes

m_2 = number of rotor phases—usually three

R = resistance per phase in rotor circuit at working temperature

$L_2\omega$ = rotor reactance per phase at standstill

$$s = \text{slip} = \frac{\text{synchronous speed} - \text{actual speed}}{\text{synchronous speed}}$$

Then
$$I_2 = \frac{sE_2}{\sqrt{R^2 + s^2L_2^2\omega^2}} \qquad (7.1)$$

$$T = \frac{sE_2^2Rm_2}{R^2 + s^2L_2^2\omega^2} \text{ in synchronous watts} \qquad (7.2)$$

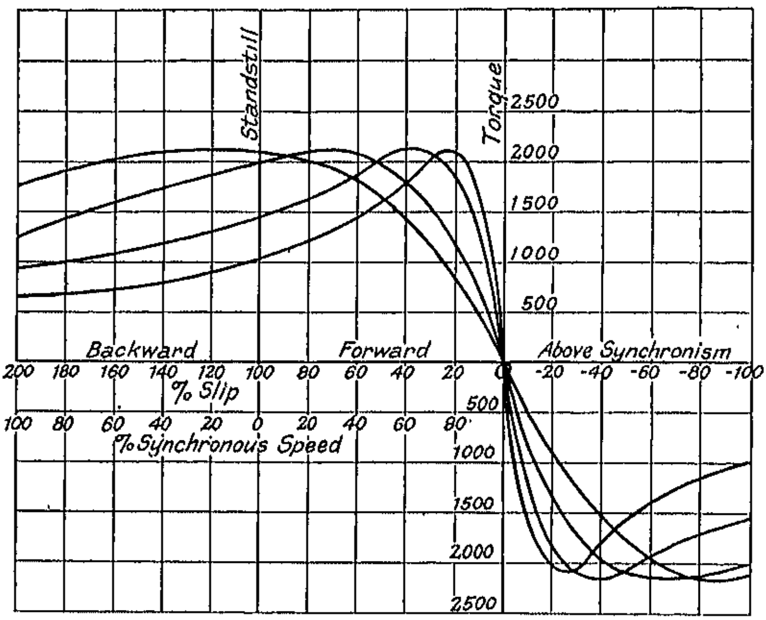


Fig. 7.1. Typical Speed/Torque Curves with Different Values of Rotor Resistance

The torque slip curves of an induction motor are given in Fig. 7.1. The slip, usually taken, is about 20 per cent in the passes. Now for *small* values of the slip

$$T \simeq \frac{m_2sE_2^2}{R} \propto \text{slip}. \qquad (7.3)$$

It will be seen, from Fig. 7.1, that the assumption that the torque is proportional to slip is sufficiently accurate for practical purposes.

$$\therefore \qquad \frac{T}{T_f} = \frac{\omega_0 - \omega}{\omega_0} \div \frac{\omega_0 - \omega_f}{\omega_0} = \frac{\omega_0 - \omega}{\omega_0 - \omega_f} \qquad (7.4)$$

Now $$T - I\frac{d\omega}{dt} = T_3 \quad . \quad . \quad . \quad . \quad (7.5)$$

since $\frac{d\omega}{dt}$ is negative.

$$\therefore \quad \frac{T_F w_0}{\omega_0 - \omega_f} - \frac{T_F \omega}{\omega_0 - \omega_f} - I\frac{d\omega}{dt} = T_3 \quad . \quad . \quad (7.6)$$

Let $$\frac{T_F}{\omega_0 - \omega_f} = a \quad . \quad . \quad . \quad . \quad (7.7)$$

Then $$a\omega_0 - a\omega - I\frac{d\omega}{dt} = T_3 \quad . \quad . \quad . \quad (7.8)$$

i.e. $$\frac{d\omega}{dt} + \frac{a\omega}{I} = \frac{a\omega_0 - T_3}{I} = \frac{c}{I} \quad . \quad . \quad . \quad (7.9)$$

where $$c = a\omega_0 - T_3 \quad . \quad . \quad . \quad . \quad (7.10)$$

Multiplying by $e^{\int \frac{a}{I} dt}$ we have

$$\frac{d}{dt}\left(\omega e^{\frac{a}{I}t}\right) = \frac{c}{I}e^{\frac{a}{I}t} \quad . \quad . \quad . \quad (7.11)$$

$$\therefore \quad \omega e^{\frac{a}{I}t} = \frac{c}{I}\int e^{\frac{a}{I}t}\, dt + E \quad . \quad . \quad . \quad (7.12)$$

where E = constant of integration

$$\therefore \quad \omega = \frac{c}{a} + Ee^{-\frac{a}{I}t} \quad . \quad . \quad . \quad . \quad (7.13)$$

When $t = 0$, $\omega = \omega_1$,

$$\therefore \quad \omega_1 = \frac{c}{a} + E \quad . \quad . \quad . \quad . \quad (7.14)$$

$$\therefore \quad E = \omega_1 - \frac{c}{a} \quad . \quad . \quad . \quad . \quad (7.15)$$

$$\therefore \quad \omega = \frac{c}{a} + \left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t} \quad . \quad . \quad . \quad (7.16)$$

$$\therefore \omega = \omega_0 - \frac{T_3}{T_f}(\omega_0 - \omega_f) + \left\{\omega_1 - \omega_0 + \frac{T_3}{T_f}(\omega_0 - \omega_f)\right\}e^{-\frac{a}{I}t} \quad (7.17)$$

This is our speed equation and from it the speed, at any time during the pass, can be determined.

At the commencement of the first pass, ω_1 will be the speed, corresponding to the friction load.

At the beginning of the second pass, it may happen that the speed has not fallen to the frictional load speed. In that case the

e end of the first interval between passes, can be found, ᵤₚeed must be substituted in the equation for the speed at ᵤginning of the second pass.

ɪ'he average speed during the pass

$$= \frac{1}{t}\int_0^t \omega dt$$

where t = time of pass

$$= \frac{1}{t}\int_0^t \left\{\frac{c}{a} + \left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t}\right\} dt \qquad (7.18)$$

$$= \frac{c}{a} + \left[-\frac{I}{a}\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t}\right]_0^t \times \frac{1}{t} \qquad (7.19)$$

$$= \frac{c}{a} - \frac{I}{a}\frac{\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}}{t} + \frac{I}{at}\left(\omega_1 - \frac{c}{a}\right) \qquad (7.20)$$

Therefore, average speed, during the pass

$$= \omega_0 - \frac{T_3}{T_f}(\omega_0 - \omega_f)$$

$$+ \frac{I}{tT_f}\left[\{\omega_0 - \omega_f\}\left\{\omega_1 - \omega_0 + \frac{T_3}{T_f}(\omega_0 - \omega_f)\right\}\right]\left[1 - e^{-\frac{a}{I}t}\right] \qquad (7.21)$$

During the interval between passes, we have

$$T = T_0 + I\frac{d\omega}{dt} \qquad (7.22)$$

Here $\frac{d\omega}{dt}$ is positive, and T_0 = frictional torque.

$$T = \frac{T_f\omega_0}{\omega_0 - \omega_f} - \frac{T_f\omega}{\omega_0 - \omega_f}$$

$$\therefore \qquad I\frac{d\omega}{dt} + a\omega = a\omega_0 - T_0 \qquad (7.23)$$

i.e.

$$\frac{d\omega}{dt} + \frac{a\omega}{I} = \frac{a\omega_0 - T_0}{I} = \frac{f}{I} \qquad (7.24)$$

where $f = a\omega_0 - T_0$

$$\therefore \qquad \omega e^{\frac{a}{I}t} = \frac{f}{I}\int e^{\frac{a}{I}t}\, dt + F \qquad (7.25)$$

When $t = t_2$ = time to the end of the first pass, $\omega = \omega_2$,

$$\therefore \qquad \omega_2 = \frac{f}{a} + Fe^{-\frac{a}{I}t_2} \qquad (7.26)$$

$$\therefore \qquad F = \left(\omega_2 - \frac{f}{a}\right) e^{\frac{a}{I} t_2} \qquad (7.27)$$

$$\therefore \qquad \omega = \frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right) e^{\frac{a}{I}(t_2 - t)} \qquad (7.28)$$

$$\therefore \quad \omega = \omega_0 - \frac{T_0}{T_f}(\omega_0 - \omega_f) + \left\{\omega_2 - \omega_0 + \frac{T_0}{T_f}(\omega_0 - \omega_f)\right\} e^{\frac{a}{I}(t_2 - t)} \qquad (7.29)$$

The average speed, during the interval,

$$= \frac{1}{t_3 - t_2}\int_{t_2}^{t_3} \omega dt = \frac{f}{a} + \frac{I}{a(t_3 - t_2)}\left(\omega_2 - \frac{f}{a}\right)\left\{1 - e^{\frac{a}{I}(t_2 - t_3)}\right\} \qquad (7.30)$$

Coming now to the question of torque, we find during the pass,

$$T = T_3 + I\frac{d\omega}{dt}. \qquad (7.31)$$

$$T = T_3 + I\left\{-\frac{a}{I}\left(\omega_1 - \frac{c}{a}\right)\right\} e^{-\frac{a}{I}t} \qquad (7.32)$$

$$= T_3 - a\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t}. \qquad (7.33)$$

$$= T_3 - \frac{T_f}{\omega_0 - \omega_f}\left\{\omega_1 - \omega_0 + \frac{T_3}{T_f}(\omega_0 - \omega_f)\right\} e^{-\frac{a}{I}t} \qquad (7.34)$$

From this the torque at any instant during the pass can be determined.

Let T_1 = torque exerted by the motor at the beginning of the pass, i.e. $t = 0$, then we have

$$T_1 = T_3 - a\left(\omega_1 - \frac{c}{a}\right) \qquad (7.35)$$

and T_2 the torque at the end of the pass $t = t_2$.

Then
$$T_2 = T_3 - a\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t_2} \qquad (7.36)$$

$$= T_3 + \frac{T_1 - T_3}{e^{\frac{a}{I}t_2}} \qquad (7.37)$$

Now
$$\frac{at_2}{I} = \frac{T_f}{\omega_0 - \omega_f} \times \frac{t_2}{I} = \frac{T_f t_2}{\omega_0\left(\frac{\omega_0 - \omega_f}{\omega_0}\right) I} = \frac{T_f t_2}{\omega_0 s I} \qquad (7.38)$$

$$\therefore \qquad T_2 = T_3 + \frac{T_1 - T_3}{e^{\frac{T_f t_2}{\omega_0 s I}}} = T_3 + \frac{T_1 - T_3}{A^{t_2}}. \qquad (7.39)$$

where
$$A = e^{\frac{T_f}{\omega_0 s I}} \qquad (7.40)$$

The torque exerted by the motor, during the interval

$$T = T_0 + I\frac{d\omega}{dt} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (7.41)$$

$$= T_0 + I\frac{d}{dt}\left\{\frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{\frac{a}{I}(t_2-t)}\right\} \qquad . \quad . \quad (7.42)$$

$$= T_0 - a\left(\omega_2 - \frac{f}{a}\right)e^{\frac{a}{I}(t_2-t)} \qquad . \quad . \quad . \quad (7.43)$$

where $t = t_2$, $T = T_2$

$$\therefore \qquad T_2 = T_0 - a\left(\omega_2 - \frac{f}{a}\right) \qquad . \quad . \quad . \quad (7.44)$$

and when $t = t_3$ the time of the beginning of the second pass $T = T_3'$

$$\therefore \qquad T_3' = T_0 - a\left(\omega_2 - \frac{f}{a}\right)e^{-\frac{a}{I}(t_3-t_2)} \qquad . \quad . \quad (7.45)$$

It is not our purpose to go into the considerations which determine the relative proportions of flywheel and motor, but it may be remarked that, where the power is generated at the works, it is important to use relatively heavy flywheels to keep the load on the generating plant as constant as possible. In those cases where power is supplied from an outside source, the method of charging affects the proportions; a heavy flywheel being necessary when the maximum demand system is in vogue, and a small flywheel and relatively large motor, when power is charged for on the flat-rate

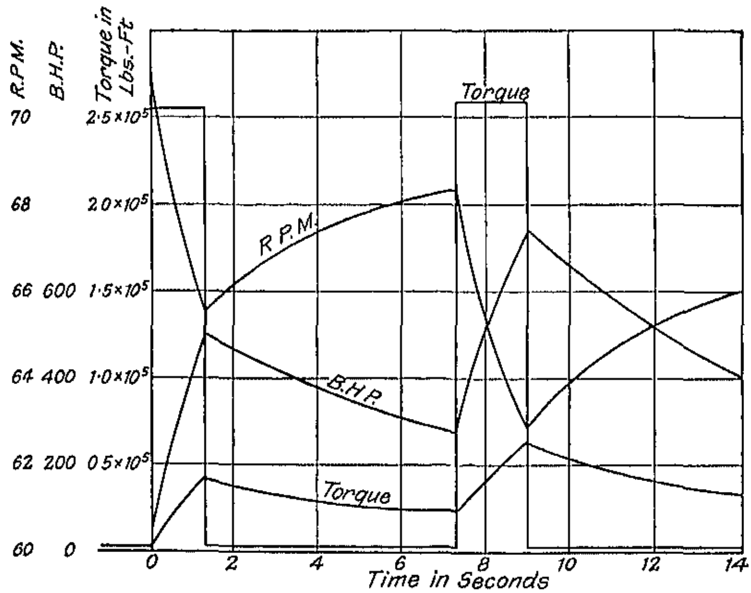


FIG. 7.2. CURVES SHOWING INDUCTION MOTORS' PERFORMANCE FOR PERMANENT RESISTANCE

system, it being important, in the latter case to reduce the friction losses as much as possible.

Application of our Formulae to a Cogging Mill

The following example will be instructive: a 20 in. to 3 ft high cogging mill for rolling ingots from 10 in. × 10 in. to 6 in. × 6 in. has six passes. The speed of the mill is 70 to 59·5 r.p.m.

The torque diagram is given in Fig. 7.2.

The flywheel employed has a moment of inertia of $0{\cdot}54 \times 10^6$ lb-ft² and has a diameter of 20 ft. The motor has a full-load output of 630 h.p. at 70 r.p.m., and the speed drop, arranged for, is from 70 to 59·5 r.p.m. The maximum torque, exerted by the motor = 83 700 lb-ft.

The slip, at 59·5 r.p.m. $= \dfrac{71{\cdot}5 - 59{\cdot}5}{71{\cdot}5} = \dfrac{12}{71{\cdot}5} = 0{\cdot}168$; percentage slip = 16·8.

At full load, the torque is 47 500 lb-ft $= \dfrac{83\,700}{1{\cdot}765}$ and the slip, at full-load torque = 9·5 per cent.

The full-load speed is, therefore, 64·7 r.p.m. and the full-load output is $\dfrac{47\,500 \times 2\pi \times 64{\cdot}7}{33\,000} = 585$ h.p.

Calculation of torque and speed in the first pass and interval

In the pass $\quad \omega = \dfrac{c}{a} + \left(\omega_1 - \dfrac{c}{a}\right) e^{\frac{a}{I}t}$

$$\frac{c}{a} = \omega_o - \frac{T_3}{T_f}(\omega_o - \omega_f)$$

$$\omega_o = 2\pi \times \frac{71{\cdot}5}{60} = 7{\cdot}48 \text{ radn/sec}$$

$$\omega_f = 2\pi \times \frac{64{\cdot}7}{60} = 6{\cdot}78 \text{ radn/sec}$$

$$T_3 = 2{\cdot}505 \times 10^5 \text{ lb-ft}$$

$$\text{friction torque} = 4000 \text{ lb-ft}$$

$$\frac{T_3}{T_f} = \frac{2{\cdot}5 \times 10^5}{4{\cdot}75 \times 10^4} = 5{\cdot}28$$

$$\omega_0 - \omega_f = 0{\cdot}7$$

$$\frac{c}{a} = 3{\cdot}78$$

no-load slip = 0·8 per cent

r.p.m., no-load = 70·9

$$\frac{a}{I} = \frac{47\,500}{0{\cdot}095 \times 7{\cdot}48 \times 0{\cdot}54 \times 10^6} = 0{\cdot}1236$$

Using equation (7.16) for the speed and equation (7.33) for the torque, we will tabulate the various quantities for the first pass—

Time in Seconds	ω	R.p.m.	Torque lb-ft	$e^{-\frac{a}{I}t}$	H.p. Output of Motor
0 (beginning)	7·40	70·9	4 000	1·000	54
0·30	7·27	69·3	13 500	0·965	179
0·60	7·14	68·0	22 500	0·928	292
0·90	7·00	66·8	31 500	0·895	409
1·27 end	6·88	65·5	41 900	0·856	524

For the first interval, we use equation (7.28) for the speed and equation (7.43) for the torque

$$\frac{f}{a} = 7{\cdot}421; \ \omega_2 - \frac{f}{a} = 6{\cdot}88 - 7{\cdot}421 = -0{\cdot}541$$

Tabulating the results for the interval, we have—

Time in Seconds	ω	R.p.m.	Torque lb-ft	$e^{-\frac{a}{I}(t_2-t)}$	H.p. of Motor
1·50	6·894	65·7	39 600	0·973	495
2·50	6·955	66·5	35 600	0·862	450
3·50	7·014	67·0	31 500	0·753	402
4·50	7·057	67·5	28 600	0·673	366
7·27	7·164	68·4	21 400	0·475	279

For the second pass, we put $\omega_1 = 7{\cdot}164$ the value of the speed at the end of the first pass.

Tabulated results, for the second pass, are as follows—

Time in Seconds	ω	R.p.m.	Torque lb-ft	$e^{-\frac{a}{I}t}$	H.p. of Motor
0	7·164	68·4	21 400	1·000	279
0·300	7·050	67·5	30 000	0·965	384
0·600	6·910	66·0	39 000	0·928	490
0·900	6·790	64·7	48 000	0·890	592
1·200	6·690	63·8	54 000	0·865	656
1·525	6·570	62·8	62 000	0·827	740

$$c = a\omega_0 - T_3$$

here $T_3 = 2{\cdot}58 \times 10^5$

$$\frac{c}{a} = \omega_0 - \frac{T_3}{T_f}(\omega_0 - \omega_f) = 7{\cdot}48 - \frac{2{\cdot}58 \times 10^5}{4{\cdot}75 \times 10^4} \times 0{\cdot}7 = 3{\cdot}68$$

$$\omega_1 - \frac{c}{a} = 7{\cdot}164 - 3{\cdot}68 = 3{\cdot}484$$

Also $$T = T_3 - a\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}$$

$$a = \frac{T_f}{\omega_0 - \omega_f} = \frac{47\ 500}{0{\cdot}7} = 67\ 855$$

In the interval of the second pass, we have

$$\omega = \frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{\frac{a}{I}(t_1 - t)}$$

and $$f = a\omega_0 - T_0$$

$$\frac{f}{a} = \omega_0 - \frac{T_0}{T_f}(\omega_0 - \omega_f) = 7{\cdot}48 - \frac{4000}{47\ 500} \times 0{\cdot}7 = 7{\cdot}421$$

$$\omega_2 = 6{\cdot}57$$

and $$\omega = 7{\cdot}421 - 0{\cdot}85\, e^{\frac{a}{I}(t_1 - t)}$$

Tabulated results for the interval after the second pass are—

Time from Commencement of Second Pass in Seconds	ω	R.p.m.	Torque lb-ft	$e^{-\frac{a}{I}(t_1 - t)}$	H.p. of Motor
2·000	6·619	63·3	58 300	0·943	700
3·000	6·718	64·4	51 700	0·827	632
4 000	6·799	64·8	46 300	0·733	570
5·000	6·871	65·5	41 200	0·646	515
6·525	7·017	67·0	31 350	0·475	399

The average speed during the first pass (from equation (7.21)) $= \omega_{av} = 7{\cdot}07$, and average speed, in revolutions per minute, in the first pass $= 67{\cdot}5$.

The motor torque, at any instant of the pass is,

$$T = T_3 - a\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t} \quad . \quad . \quad . \quad (7{\cdot}46)$$

and the average torque, during the pass,

$$= \frac{1}{t}\int_0^t T\,dt$$

$$= \frac{1}{t}\int_0^t \left\{T_3 - a\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}\right\}dt$$

$$= \frac{1}{t}\left[T_3 t + I\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}\right]_0^t$$

$$= T_3 + \frac{I}{t}\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t} - \frac{I}{t}\left(\omega_1 - \frac{c}{a}\right) \quad . \quad . \quad (7{\cdot}47)$$

For the first pass

$$\text{av. } T = 2{\cdot}505 \times 10^6 + (0{\cdot}54 \times 10^6 \times 3{\cdot}62 \times 0{\cdot}855 - 0{\cdot}54 \times 10^6 \times 3{\cdot}62) \times \frac{1}{1{\cdot}27}$$

$$= 27\,500 \text{ lb-ft}$$

The results, for the two passes, are shown in Fig. 7.2.

Constant Output in the Passes

The power equation for the pass is

$$T\omega + I\frac{d\omega}{dt}\omega = W \quad . \quad . \quad . \quad (7{\cdot}48)$$

where W = constant power in the pass

As before $\qquad T = a\omega_0 - a\omega$

$$\therefore \quad a\omega\omega_0 - a\omega^2 - I\frac{d\omega}{dt}\omega = W \quad . \quad . \quad . \quad (7{\cdot}49)$$

$$\therefore \quad dt = \frac{I\omega d\omega}{a\omega\omega_0 - a\omega^2 - W} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7{\cdot}50)$$

$$= -\frac{1}{2a} \times \frac{I(a\omega_0 - 2a\omega)d\omega}{a\omega\omega_0 - a\omega^2 - W} + \frac{I}{2}\frac{\omega_0 d\omega}{(a\omega\omega_0 - a\omega^2 - W)} \quad (7{\cdot}51)$$

$$\therefore \; t = -\frac{I}{2a}\log(a\omega\omega_0 - a\omega^2 - W) + \frac{I\omega_0}{2}\int\frac{d\omega}{a\omega\omega_0 - a\omega^2 - W} + C \quad . \quad . \quad (7{\cdot}52)$$

$$\therefore \; t = -\frac{I}{2a}\log(a\omega\omega_0 - a\omega^2 - W) - \frac{I\omega_0}{2a\sqrt{\dfrac{W}{a} - \dfrac{\omega_0^2}{4}}}\tan^{-1}\left(\frac{\omega - \frac{1}{2}\omega_0}{\sqrt{\dfrac{W}{a} - \dfrac{\omega_0^2}{4}}}\right) + C \quad (7{\cdot}53)$$

When $t = 0$, $\omega = \omega_1$,

$$\therefore \; 0 = -\frac{I}{2a}\log(a\omega_0\omega_1 - a\omega_1^2 - W) - \frac{I\omega_0}{2a\sqrt{\dfrac{W}{a} - \dfrac{\omega_0^2}{4}}}\tan^{-1}\left(\frac{\omega_1 - \frac{1}{2}\omega_0}{\sqrt{\dfrac{W}{a} - \dfrac{\omega_0^2}{4}}}\right) + C \quad (7{\cdot}54)$$

$$\therefore \; C = \frac{I}{2a}\log(a\omega_0\omega_1 - a\omega_1^2 - W) + \frac{I\omega_0}{2a\sqrt{\dfrac{W}{a} - \dfrac{\omega_0^2}{4}}}\tan^{-1}\left(\frac{\omega_1 - \frac{1}{2}\omega_0}{\sqrt{\dfrac{W}{a} - \dfrac{\omega_0^2}{4}}}\right) \quad (7{\cdot}55)$$

$$\therefore\ t = \frac{I}{2a}\log\left(\frac{(a\omega_0\omega_1 - a\omega_1^2 - W)}{a\omega_0\omega - a\omega^2 - W}\right)$$

$$+ \frac{I\omega_0}{2a\sqrt{\frac{W}{a} - \frac{\omega_0^2}{4}}}\left[\tan^{-1}\left(\frac{\omega_1 - \frac{1}{2}\omega_0}{\sqrt{\frac{W}{a} - \frac{\omega_0^2}{4}}}\right) - \tan^{-1}\left(\frac{\omega - \frac{1}{2}\omega_0}{\sqrt{\frac{W}{a} - \frac{\omega_0^2}{4}}}\right)\right] \quad (7.56)$$

In passes, the last term is usually negligibly small and, where it is not negligible, the inverse tangent can be expanded and an approximation obtained.

$$\therefore \qquad t \simeq \frac{I}{2a}\log\left(\frac{a\omega_0\omega_1 - a\omega_1^2 - W}{a\omega_0\omega - a\omega^2 - W}\right) \quad . \quad . \quad (7.57)$$

Put $a\omega_0\omega_1 - a\omega_1^2 - W = d$

then
$$\frac{d}{a\omega_0\omega - a\omega^2 - W} = e^{+\frac{2at}{I}} \quad . \quad . \quad . \quad (7.58)$$

$$\therefore \qquad de^{-\frac{2at}{I}} = a\omega_0\omega - a\omega^2 - W \quad . \quad . \quad . \quad (7.59)$$

i.e.
$$a\omega^2 - a\omega_0\omega + W + de^{-\frac{2at}{I}} = 0 \quad . \quad . \quad (7.60)$$

$$\therefore \qquad \omega = \frac{a\omega_0 \pm \sqrt{a^2\omega_0^2 - 4(W + de^{-\frac{2at}{I}})a}}{2a} \quad . \quad . \quad (7.61)$$

Equation (7.61) gives the speed at any time t from the commencement of the pass.

The output of the induction motor, during the pass,

$$= W + I\omega\frac{d\omega}{dt} \quad . \quad . \quad . \quad . \quad (7.62)$$

and
$$\frac{d\omega}{dt} = \frac{2ade^{-\frac{2at}{I}}}{I\sqrt{a^2\omega_0^2 - 4\left(W + de^{-\frac{2at}{I}}\right)a}} \quad . \quad . \quad (7.63)$$

The motor output can thus be determined for constant total output in the passes.

In the interval between the passes, the power equation is

$$I\frac{d\omega}{dt}\omega + W_0 = T\omega \quad . \quad . \quad . \quad (7.64)$$

where W_0 = frictional power, which is assumed to remain constant.

The Case of the Automatic Slip Regulator

We will now consider the case of a rolling-mill cycle, in which an automatic slip regulator is introduced at a predetermined value of the load. An induction motor, of the slip-ring type, whose stator

is excited from the secondary circuit of a series transformer, the primary of which is inserted in the leads to the motor, is used for operating the electrodes. The movement of the rotor is resisted by means of a spring. As the current in the leads to the stator of the main motor increases, the current in the induction motor of the slip regulator increases. The torque varies as the square of the current, so the displacement of the rotor will vary as the square of the current. A lever, attached to the rotor of the regulator motor, moves the blades of the liquid resistance switch in and out of the liquid, thus varying the resistance in the main motor circuit. An increase in load causes the blades to leave the liquid partially, and thus to increase the resistance and slip of the main motor. We will investigate shortly the law, according to which such regulators should be designed, to meet the objects in view. Unlike the case of permanent resistance, the torque of the induction motor remains constant, or, at least, this is the condition to be aimed at, since the current to the motor remains constant, if this condition is fulfilled.

Taking the case of a continuous mill, with constant total torque in the pass, our torque equation is—

$$T - I\frac{d\omega}{dt} = T_3 \quad . \quad . \quad . \quad . \quad (7{\cdot}65)$$

T_3 = constant total torque in the pass, and the other symbols have the same significance as before.

$$\frac{d\omega}{dt} = \frac{T - T_3}{I}$$

$$\therefore \qquad \omega = \left(\frac{T - T_3}{I}\right) t + G \quad . \quad . \quad . \quad (7.66)$$

When $t = t_1$, the time when the slip regulator comes into action, let $\omega = \omega_1$,

$$\therefore \qquad \omega_1 = \left(\frac{T - T_3}{I}\right) t_1 + G \quad . \quad . \quad . \quad (7.67)$$

$$\therefore \qquad G = \omega_1 - \left(\frac{T - T_3}{I}\right) t_1 \quad . \quad . \quad . \quad (7.68)$$

and

$$\omega = \omega_1 + \frac{T - T_3}{I}(t - t_1) \quad . \quad . \quad . \quad (7.69)$$

ω_1 is easily determined, for it is simply the angular velocity, in radians per second, when the regulator comes in.

It will be seen that the relation between angular velocity and time is linear.

In the interval between passes, our torque equation is

$$T - I\frac{d\omega}{dt} = T_0 \quad . \quad . \quad . \quad . \quad (7.70)$$

where T_0 = friction torque

$$\therefore \qquad \omega = \left(\frac{T - T_0}{I}\right) t + F \qquad . \quad . \quad . \quad (7.71)$$

When $t = t_2$, the time to the end of the first pass, $\omega = \omega_2$,

$$\therefore \qquad \omega_2 = \left(\frac{T - T_0}{I}\right) t_2 + F \qquad . \quad . \quad . \quad (7.72)$$

$$\therefore \qquad F = \omega_2 - \left(\frac{T - T_0}{I}\right) t_2 \qquad . \quad . \quad . \quad (7.73)$$

Therefore, ω in the interval

$$= \omega_2 + \left(\frac{T - T_0}{I}\right)(t - t_2) \qquad . \quad . \quad . \quad (7.74)$$

The power of the motor, at any time during the pass, with the regulator in

$$= T\omega = T\omega_1 + T\left(\frac{T - T_3}{I}\right)(t - t_1) \qquad . \quad . \quad (7.75)$$

The average speed in the pass, during the time the regulator is in action

$$= \tfrac{1}{2}(\omega_1 + \omega_2)$$

where ω_1 = angular speed at the time of entrance of the regulator

and $\quad \omega_2$ = angular speed at the end of the pass

$$= \tfrac{1}{2}\left\{\omega_1 + \omega_1 + \frac{T - T_3}{I}(t_2 - t_1)\right\} \qquad . \quad . \quad (7.76)$$

$$= \omega_1 + \frac{T - T_3}{I}\,\frac{(t_2 - t_1)}{2} \qquad . \quad . \quad . \quad (7.77)$$

In the interval, the torque equation is—

$$T - I\frac{d\omega}{dt} = T_0 \qquad . \quad . \quad . \quad . \quad (7.78)$$

where T_0 = frictional torque

It is interesting to enquire what values of rotor resistance are required in order that the torque of the motor shall remain constant. The torque, in synchronous watts,

$$= \frac{m_2 E_2^2 R_2 s}{R_2^2 + s^2 L_2^2 \omega_0^2}$$

If T = motor torque in *kilogramme-metres*, then

$$T\omega_s 9{\cdot}81 = \frac{m_2 E_2^2 R_2 s}{R_2^2 + s^2 L_2^2 \omega_0^2} \qquad . \quad . \quad . \quad (7.79)$$

where m_2 = number of rotor phases

R_2 = rotor resistance per phase + external resistance per phase

E_2 = volts per rotor phase at standstill

$$\therefore \quad T\omega_s 9{\cdot}81 R_2^2 - m_2 E_2^2 s R_2 + T\omega_s 9{\cdot}81 s^2 L_2^2 \omega_0^2 = 0 \quad . \quad (7.80)$$

$$\therefore \quad R_2 = \frac{m_2 E_2^2 s \pm s\sqrt{m_2^2 E_2^4 - 386 T^2 \omega_0^2 \omega_s^2 L_2^2}}{19{\cdot}62\, T\omega_s} \quad . \quad (7.81)$$

where $\omega_0 = 2\pi \times$ supply frequency

ω_s = synchronous angular velocity of the rotor in radians per second

$$\therefore \quad \frac{R_2}{s} = \frac{m_2 E_2^2 \pm \sqrt{m_2^2 E_2^4 - 386 T^2 \omega_0^2 \omega_s^2 L_2^2}}{19{\cdot}62\, T\omega_s} \quad . \quad (7.82)$$

Since $$s = \frac{\omega_s - \omega}{\omega_s}$$

$$R_2 = \frac{\omega_s - \omega}{\omega_s}\left\{\frac{m_2 E_2^2 \pm \sqrt{m_2^2 E_2^4 - 386 T^2 \omega_0^2 \omega_s^2 L_2^2}}{19{\cdot}62\, T\omega_s}\right\} \quad (7.83)$$

$\therefore$ R_2 in the pass

$$= \frac{-\omega_1 - \left(\frac{T_1 - T_3}{I}\right)(t - t_1) + \omega_s}{\omega_s}$$

$$\times \left\{\frac{m_2 E_2^2 \pm \sqrt{m_2^2 E_2^4 - 385 T^2 \omega_0^2 \omega_s^2 L_2^2}}{19{\cdot}62\, T\omega_s}\right\} \quad (7.84)$$

Equation (7.84) gives the value of the total resistance per phase, required for constant torque, at any time during the pass.

Note. Since

$$\frac{p}{2} \times \text{revolutions per second (synchronous)} = \text{frequency} = f$$

$$\text{revolutions per second (synchronous)} = \frac{2f}{p}$$

where p = poles

$$\omega_s = 2\pi \times \text{synchronous revolutions per second} = \frac{4\pi f}{p}$$

$$\therefore \quad \omega_s = \frac{2\omega_0}{p} \quad . \quad . \quad . \quad . \quad (7.85)$$

It is clear the resistance should be directly proportional to the slip for constant torque.

Continuous Mill with Automatic Slip Regulator and Constant Output during the Pass

The power equation is—

$$T\omega - I\frac{d\omega}{dt}\omega = W \quad . \quad . \quad . \quad (7.86)$$

W = constant output during the pass.

$$\therefore \qquad \frac{I\omega d\omega}{T\omega - W} = dt \quad . \quad . \quad . \quad . \quad (7.87)$$

$$\frac{I}{W}\left\{\frac{W}{T}\left(\frac{1}{\frac{T}{W}\omega - 1} + 1\right)\right\}d\omega = dt \quad . \quad . \quad (7.88)$$

$$\therefore \qquad C + \frac{IW}{T^2}\log\left(\frac{T}{W}\omega - 1\right) + \frac{I}{T}\omega = t \quad . \quad . \quad (7.89)$$

To determine C, put $t = t_1$, then $\omega = \omega_1$, the speed at time t_1, when the regulator comes in—

$$C + \frac{IW}{T^2}\log\left(\frac{T}{W}\omega_1 - 1\right) + \frac{I}{T}\omega_1 = t_1 \quad . \quad . \quad (7.90)$$

$$\therefore \qquad C = -\frac{IW}{T^2}\log\left(\frac{T}{W}\omega_1 - 1\right) + t_1 - \frac{I}{T}\omega_1 \quad . \quad (7.91)$$

$$\therefore \qquad t = \frac{IW}{T^2}\log\frac{\left(\frac{T}{W}\omega - 1\right)}{\left(\frac{T}{W}\omega_1 - 1\right)} + \frac{I}{T}(\omega - \omega_1) + t_1 \quad . \quad (7.92)$$

$$\therefore \quad t = \frac{IW}{T^2}\log\left\{1 + \frac{\frac{T}{W}(\omega - \omega_1)}{\frac{T}{W}\omega_1 - 1}\right\} + \frac{I}{T}(\omega - \omega_1) + t_1 \quad (7.93)$$

Expanding the log, we have—

$$\log\left\{1 + \frac{\frac{T}{W}(\omega - \omega_1)}{\frac{T}{W}\omega_1 - 1}\right\}$$

$$= \frac{\frac{T}{W}(\omega - \omega_1)}{\frac{T}{W}\omega_1 - 1} - \frac{1}{2}\left\{\frac{\frac{T}{W}(\omega - \omega_1)}{\frac{T}{W}\omega_1 - 1}\right\}^2 + \ldots \text{ etc.}$$

The second and higher powers are negligibly small.

$$\therefore \qquad t \simeq \frac{IW}{T^2}\left\{\frac{\frac{T}{W}(\omega - \omega_1)}{\frac{T}{W}\omega_1 - 1}\right\} + \frac{I}{T}(\omega - \omega_1) + t_1 \quad . \quad (7.94)$$

and
$$\omega = \omega_1 + \left(\frac{T}{I} - \frac{W}{I\omega_1}\right)(t - t_1) \quad . \quad . \quad (7.95)$$

If, as is sometimes the case, $\frac{IW}{T^2}$ is fairly large, it may be necessary to take the second, or even the third power of the expansion of the log, and these will give a quadratic and a cubic equation in ω, which can be solved by the usual methods.

Example on Steel Tube Mill

This is an admirable case of the application of the automatic slip regulator, for the passes and intervals are relatively long.

The cycle consists of one pass of 45 sec and one interval of 75 sec.

The total torque required in the pass = 26 000 lb-ft.

The friction torque = 2100 lb-ft.

The speed of the mill is 150 r.p.m.

The motor is geared to the mill shaft, and the reduction ratio is 3·23.

The total torque-time in lb-ft/sec = 26 000 × 45 + 2100 × 75
= 1·327 × 10⁶

$$= 1{\cdot}327 \times 10^6$$

The average torque in the cycle $= \frac{1{\cdot}327 \times 10^6}{120}$

$$= 11{\cdot}05 \times 10^3 \text{ lb-ft}$$

To find the size of the flywheel, we proceed as follows—

Let I = moment of inertia of the flywheel in lb-ft²

ω_1 = angular velocity, in radians per second, at the beginning of the pass

ω_2 = angular velocity, in radians per second, at the end of the pass

The energy given out, by the wheel, in the pass

$$= \tfrac{1}{2}I(\omega_1^2 - \omega_2^2)$$

where ω_1 = 51·8 radn/sec = 495 r.p.m.

$$\omega_2 = 2\pi \times \frac{450}{60} = 47{\cdot}1 \text{ radn/sec}$$

The torque given out by the flywheel in the pass

$$= \frac{26\,000 - 11\,050}{3{\cdot}23} = 4640 \text{ lb-ft}$$

$$\therefore \qquad \tfrac{1}{2}I(51{\cdot}8^2 - 47{\cdot}1^2) = 4640 \times 2\pi \times \frac{475}{60} \times 45$$

$$\therefore \qquad I = 45\,000 \text{ lb-ft}^2$$

Average horse-power of the motor

$$= \frac{11\,050 \times 2\pi \times 474}{3{\cdot}23 \times 33\,000 \times 0{\cdot}95} = 326 \text{ h.p.}$$

This, of course, will not be the rating of the motor. The motor will be required to be put in for the r.m.s. brake-horse-power over the cycle. We have seen that the speed line is a straight line, after the regulator comes in. Before its introduction, however, the motor is working with permanent resistance in its circuit, viz. its own rotor resistance. Applying our equations, we have

$$\omega = \frac{c}{a} + \left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}$$

$$\frac{c}{a} = \omega_0 - \frac{T_3}{a}$$

$$\omega_0 = 500 \times \frac{2\pi}{60} = 52{\cdot}4 \text{ radn/sec}$$

$$T_3 = \frac{26\,000}{3{\cdot}23} = 8050$$

$$a = \frac{T_f}{\omega_0 - \omega_f} = \frac{11\,050 \times 60}{3{\cdot}23 \times 2\pi \times 15} = 2180$$

Natural slip at full load = 3 per cent.

$$\frac{c}{a} = 52{\cdot}4 - \frac{8050}{2180} = 52{\cdot}4 - 3{\cdot}69 = 48{\cdot}71$$

$$\frac{a}{I} = \frac{2180}{45\,000} = 0{\cdot}0485$$

The time to reach full-load torque is given by the equation—

$$50{\cdot}85 = 48{\cdot}71 + 3{\cdot}09e^{-\frac{a}{I}t}$$

$$\therefore \qquad t = 7{\cdot}6 \text{ sec}$$

for $\qquad 50{\cdot}85 = 48{\cdot}71 + 3{\cdot}09e^{-0{\cdot}0485t}$

The torque equation, for the beginning of the pass up to the point when the regulator comes in is—

$$T = T_3 - a\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t}$$
$$= 8050 - 2180(51{\cdot}8 - 48{\cdot}71)e^{-0{\cdot}0485t}$$
$$= 8050 - 6745e^{-0{\cdot}0485t}$$

The results, for the pass, are tabulated below—

Time in Seconds	ω	R.p.m.	$e^{-0\cdot0485t}$	Torque lb-ft	H.p. of Motor
0	51·80	495·0	1·000	1305	122·7
2·0	51·50	492·0	0·905	1950	182·5
4·0	51·25	490·0	0·822	2520	235·1
6·0	51·01	487·5	0·746	3025	281·1
7·5	50·84	485·0	0·692	3390	311·0
Reg. comes in 45	47·91	457·0	—	3390	295·0

When the regulator is in action, the speed falls in a linear fashion and

$$\omega = \omega_1 + \frac{T - T_3}{I}(t - t_1)$$

therefore, at the end of the pass

$$\omega = 51{\cdot}8 + \frac{3390 - 8050}{45\,000} \times 37{\cdot}5 = 47{\cdot}91$$

In the interval, the speed rises in a straight line, the torque remaining constant, till the natural torque-slip curve of the motor is

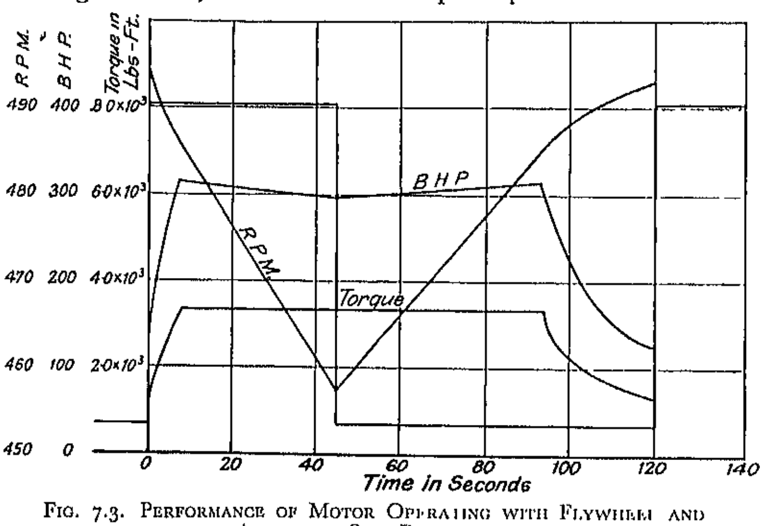


FIG. 7.3. PERFORMANCE OF MOTOR OPERATING WITH FLYWHEEL AND AUTOMATIC SLIP REGULATOR

reached. This is seen in Fig. 7.3. Any further increase in speed, beyond that corresponding to the natural slip of the motor, results in a reduction in torque and our speed and torque equations, from this point, are simply those for permanent resistance, i.e. the resistance of the rotor windings, viz. equation (7.28) and (7.43).

In the interval, the speed rises according to the equation

$$\omega = \omega_2 + \frac{T - T_0}{I}(t - t_2)$$

$$\therefore \qquad \frac{d\omega}{dt} = \frac{T - T_0}{I} = \frac{3390 - 650}{45\ 000} = 0{\cdot}061$$

The speed of the motor, at full-load torque,

$$= 2\pi \times \frac{485}{60} = 50{\cdot}84 \text{ radn/sec}$$

Increase in speed from the *end* of the pass

$$= 50{\cdot}84 - 47{\cdot}91 = 2{\cdot}93 \text{ radn/sec}$$

Therefore, time to reach the natural torque-slip curve

$$= \frac{2{\cdot}93}{0{\cdot}061} = 48 \text{ sec}$$

The remaining 27 sec is taken up on the permanent resistance of the rotor.

The equation

$$\omega = \frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{\frac{a}{I}(t_2 - t)}$$

applies to this period.

$$\frac{f}{a} = \omega_0 - \frac{T_0}{a} = 52{\cdot}25 - \frac{650}{2180} = 51{\cdot}954$$

$$\omega_2 = 50{\cdot}84$$

$$\omega_2 - \frac{f}{a} = -1{\cdot}114$$

Tabulating the results for the 27-sec interval, we have—

Time *from* 48 *sec* in interval	ω	R.p.m.	$e^{0\cdot0185t}$	Torque	H.p.
2	50·950	486·0	0·905	2850	263·5
4	51·024	488·0	0·822	2650	264·0
6	51·117	489·0	0·746	2460	229·0
10	51·270	490·0	0·610	2130	198·5
14	51·380	490·3	0·510	1890	176·5
17	51·465	491·0	0·433	1700	158·5
27	51·648	493·0	0·275	1318	124·0

The corresponding torque in this period of 27 sec is

$$T = T_0 - a\left(\omega_2 - \frac{f}{a}\right) e^{\frac{a}{I}(t_4 - t)}$$

$$= 650 - 2180(-1\cdot114)e^{\frac{a}{I}(t_4-t)}$$

$$= 650 + 2430e^{-0\cdot0186(t_4-t)}$$

These results are plotted in Fig. 7.3.

R.M.S. H.P.

Let t_1 = time in pass, until regulator comes into action

t_2 = time till the end of the pass, from beginning of pass

t_3 = time in interval that the motor is on the regulator

t_4 = time till end of cycle

The power, in the period t_1,

$$= \left\{\frac{c}{a} + \left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}\right\}\left\{T_3 - a\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}\right\} = W$$

$$= T_3\frac{c}{a} + T_3\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t} - c\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}$$

$$- a\left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t} = W \quad . \quad . \quad . \quad . \quad . \quad (7\cdot96)$$

$$W^2 = T_3^2\frac{c^2}{a^2} + 2T_3^2\frac{c}{a}\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t} - 2T_3\frac{c^2}{a}\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}$$

$$- 2T_3c\left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t} + T_3^2\left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t}$$

$$- 2T_3a\left(\omega_1 - \frac{c}{a}\right)^3 e^{-\frac{3a}{I}t} - T_3c\left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t}$$

$$+ 2ac\left(\omega_1 - \frac{c}{a}\right)^3 e^{-\frac{3a}{I}t} + c^2\left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t}$$

$$+ a^2\left(\omega_1 - \frac{c}{a}\right)^4 e^{-\frac{4a}{I}} \quad . \quad . \quad . \quad . \quad . \quad (7\cdot97)$$

$$\int_0^{t_1} W^2 dt = T_3{}^2 \frac{c^2}{a^2} t_1 - 2 T_3{}^2 \frac{c}{a^2} I \left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I} t_1}$$

$$+ 2 T_3{}^2 \frac{c}{a^2} I \left(\omega_1 - \frac{c}{a}\right) + 2 T_3 \frac{c^2}{a^2} I \left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I} t_1}$$

$$- 2 T_3 \frac{c^2}{a^2} I \left(\omega_1 - \frac{c}{a}\right) + \frac{T_3 c I}{a} \left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I} t_1}$$

$$- \frac{T_3 c I}{a} \left(\omega_1 - \frac{c}{a}\right)^2 - \frac{T_3{}^2 I}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I} t_1}$$

$$+ \frac{T_3{}^2 I}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 + \frac{2}{3} T_3 I \left(\omega_1 - \frac{c}{a}\right)^3 e^{-\frac{3a}{I} t}$$

$$- \frac{2}{3} T_3 I \left(\omega_1 - \frac{c}{a}\right)^3 + \frac{T_3 c I}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I} t_1}$$

$$- \frac{T_3 c I}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 - \frac{2}{3} Ic \left(\omega_1 - \frac{c}{a}\right)^3 e^{-\frac{3a}{I} t_1}$$

$$+ \frac{2}{3} Ic \left(\omega_1 - \frac{c}{a}\right)^3 - \frac{Ic^2}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I} t_1}$$

$$+ \frac{Ic^2}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 - \frac{aI}{4} \left(\omega_1 - \frac{c}{a}\right)^4 e^{-\frac{4a}{I} t_1} + \frac{aI}{4} \left(\omega_1 - \frac{c}{a}\right)^4 \quad . \quad . \quad (7.98)$$

The power in time

$$t_2 - t_1 = T\omega_1 + T\left(\frac{T - T_3}{I}\right)(t - t_1) = W_2 \quad . \quad (7.99)$$

$$= T\omega_1 + \frac{T^2 - TT_3}{I}(t - t_1) = W_2 \quad . \quad (7.100)$$

$$W_2{}^2 = T^2\omega_1{}^2 + 2T\left(\frac{T^2 - TT_3}{I}\right)\omega_1(t - t_1)$$

$$+ \left(\frac{T^2 - TT_3}{I}\right)^2 \{t^2 - 2tt_1 + t_1{}^2\} \quad . \quad (7.101)$$

$$\int_{t_1}^{t_2} W_2^2 dt = T^2\omega_1^2(t_2 - t_1) + \omega_1 2T\left(\frac{T^2 - T_1T_3}{I}\right)\left[\tfrac{1}{2}t^2 - t_1 t\right]_{t_1}^{t_2}$$
$$+ \left(\frac{T^2 - TT_3}{I}\right)^2 \left\{\frac{t^3}{3} - t^2t_1 + t_1^2 t\right\}_{t_1}^{t_2} \quad . \quad (7.102)$$

$$= T^2\omega_1^2(t_2 - t_1) + 2T\omega_1\left(\frac{T^2 - TT_3}{I}\right)\left[\tfrac{1}{2}t_2^2 + \tfrac{1}{2}t_1^2 - t_1t_2\right]$$
$$+ \left(\frac{T^2 - TT_3}{I}\right)^2\left[\frac{t_2^3}{3} - \frac{t_1^3}{3} - t_2^2t_1 + t_1^2t_2\right] \quad . \quad (7.103)$$

A corresponding expression for the interval $t_3 - t_2$ as this applies, and a similar expression to the first for the last part of the cycle. The speed in the third part of the cycle

$$\omega = \omega_2 + \left(\frac{T - T_0}{I}\right)(t - t_2) \quad . \quad . \quad (7.104)$$

and
$$W_3 = T\omega = T\left[\omega_2 + \left(\frac{T - T_0}{I}\right)(t - t_2)\right] \quad . \quad (7.105)$$

and

$$\int_{t_2}^{t_3} W_3^2 dt = T^2\omega_2^2(t_3 - t_2) + 2T^2\left(\frac{T - T_0}{I}\right)\omega_2\left[\tfrac{1}{2}t_3^2 + \tfrac{1}{2}t_2^2 - t_3t_2\right]$$
$$+ \left(\frac{T^2 - TT_0}{I}\right)^2\left[\frac{t_3^3}{3} - \frac{t_2^3}{3} - t_3^2t_2 + t_2^2t_3\right] \quad . \quad (7.106)$$

Again for the last period

$$\omega = \frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{-\frac{a}{I}t} \quad . \quad . \quad . \quad (7.107)$$

and the torque
$$= T_0 - a\left(\omega_2 - \frac{f}{a}\right)e^{-\frac{a}{I}t}$$

and
$$W'_4 = T\omega$$

$$\int_{t_3}^{t_4} W'^2_4 dt = \int_{t_3}^{t_4}\left[\left\{T_0 - a\left(\omega_2 - \frac{f}{a}\right)e^{-\frac{a}{I}t}\right\}\left\{\frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{-\frac{a}{I}t}\right\}\right]^2 dt$$
$$. \quad . \quad (7.108)$$

It will be noticed that this is the same product, in form, as for the first period, and one should substitute, in the expression for the first period, T_0 for T_3, $\frac{f}{a}$ for $\frac{c}{a}$, and ω_2 for ω_1.

The r.m.s. power

$$= \sqrt{\frac{\int_0^{t_1} W_1^2 dt + \int_{t_1}^{t_2} W_2^2 dt + \int_{t_2}^{t_3} W'^2_3 dt + \int_{t_3}^{t_4} W_4^2 dt}{t_4}} \quad . \quad (7.109)$$

It is simpler, in any given case, to square the ordinates of power at each point, and find the area with the planimeter. Then find the mean of the squared ordinates and take the square root.

The Ward-Leonard System

Consider next the case of the electric winder, operating on the Ward-Leonard system. The induction motor is provided with a flywheel on the shaft, and an automatic slip-regulator in the rotor circuits.

Fig. 7.4 gives the winding diagram for an electric hoist.

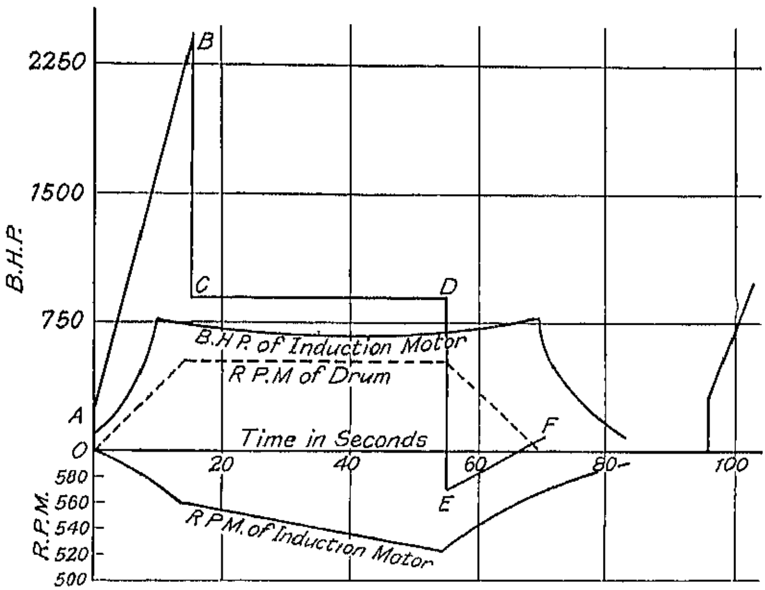


FIG. 7.4

Let W = output at B

t_1 = period of acceleration in seconds

Then $$T\omega - I\omega \frac{d\omega}{dt} = W \times \frac{t}{t_1} \quad . \quad . \quad . \quad (7.110)$$

and $$\frac{T}{T_f} = \frac{\omega_0 - \omega}{\omega_0 - \omega_f}; \quad \therefore \quad T = \frac{T_f\omega_0}{\omega_0 - \omega_f} - \frac{T_f\omega}{\omega_0 - \omega_f} \quad . \quad (7.111)$$

$$\therefore \quad a_1\omega - a_2\omega^2 - \omega \frac{d\omega}{dt} = c_1 t \quad . \quad . \quad . \quad (7.112)$$

where $c_1 = \frac{W}{It_1}$

$$a_1 = \frac{T_f\omega_0}{(\omega_0 - \omega_f)I}; \quad a_2 = \frac{T_f}{(\omega_0 - \omega_f)I} \quad . \quad . \quad (7.113)$$

The solution of equation (7.112) may be obtained, approximately, thus—

Let $\omega = a$ when $t = 0$, and let $\omega = ue^{-a_2 t}$, so that $u = a$ when $t = 0$.

$$\frac{d\omega}{dt} = \frac{du}{dt} e^{-a_2 t} - a_2 u e^{-a_2 t} \quad . \quad . \quad . \quad (7.114)$$

$$\therefore \qquad \omega \frac{d\omega}{dt} = u \frac{du}{dt} e^{-2a_2 t} - a_2 u^2 e^{-2a_2 t} \quad . \quad . \quad (7.115)$$

$$\therefore \qquad a_1 u e^{-a_2 t} - a_2 u^2 e^{-2a_2 t} - u \frac{du}{dt} e^{-2a_2 t} + a_2 u^2 e^{-2a_2 t} = c_1 t \quad . \quad (7.116)$$

$$\therefore \qquad a_1 u e^{-a_2 t} - u \frac{du}{dt} e^{-2a_2 t} = c_1 t \quad . \quad . \quad . \quad (7.117)$$

$$\therefore \qquad a_1 u e^{-a_2 t} - c_1 t = u \frac{du}{dt} e^{-2a_2 t} \quad . \quad . \quad . \quad (7.118)$$

and

$$\frac{du}{dt} = a_1 e^{a_2 t} - \frac{c_1}{u} e^{2a_2 t} \times t \quad . \quad . \quad . \quad (7.119)$$

Expand u in a series of Maclaurin

$$u = f(t) = f(o) + tf'(o) + \frac{t^2}{\underline{|2}} f''(o) + \dots \quad . \quad (7.120)$$

$$f(o) = a$$

$$f'(t) = \frac{du}{dt} = a_1 e^{a_2 t} - \frac{c_1}{u} e^{2a_2 t} \times t \quad . \quad . \quad (7.121)$$

$$\therefore \qquad f'(o) = a_1$$

$$f''(t) = a_1 a_2 e^{a_2 t} - \frac{c_1}{u} e^{2a_2 t} - \frac{2a_2 c_1}{u} e^{2a_2 t} \times t + \frac{c_1 t e^{2a_2 t}}{u^2}\left(a_1 e^{a_2 t} - \frac{c_1}{u} e^{2a_2 t}\right) \quad . \quad . \quad (7.122)$$

$$f''(o) = a_1 a_2 - \frac{c_1}{a} \quad . \quad . \quad . \quad (7.123)$$

$$\therefore \qquad \omega = ue^{-a_2 t} \quad . \quad . \quad . \quad . \quad (7.124)$$

$$= e^{-a_2 t}\left\{a + a_1 t - \tfrac{1}{2}\left(\frac{c_1}{a} - a_1 a_2\right) t^2 + \dots\right\} \quad . \quad (7.125)$$

This expression represents the manner in which the speed of the induction motor falls, *during the accelerating stage of the winding cycle of the d.c. motor.*

We will find the speed and output of the induction motor during the remainder of the cycle.

The power equation is

$$T\omega - I\frac{d\omega}{dt}\omega = Ct \quad . \quad . \quad . \quad (7.126)$$

where C = rate of increase of power with time on the accelerating part of the curve (i.e. the part AB in Fig. 7.4.).

In this part of the cycle, the torque of the motor is constant

$$\therefore \qquad \omega\frac{d\omega}{dt} - \frac{T\omega}{I} + \frac{C}{I}t = 0 \quad . \quad . \quad . \quad (7.127)$$

Let $\omega = vt$.

$$\frac{d\omega}{dt} = t\frac{dv}{dt} + v$$

then equation (7.127) becomes

$$vt^2\frac{dv}{dt} + v^2t - \frac{T}{I}vt + \frac{C}{I}t = 0 \quad . \quad . \quad (7.128)$$

$$vt^2\frac{dv}{dt} + t\left(v^2 - \frac{T}{I}v + \frac{C}{I}\right) = 0$$

i.e.

$$-\frac{vdv}{v^2 - \frac{T}{I}v + \frac{C}{I}} = \frac{dt}{t} \quad . \quad . \quad . \quad (7.129)$$

Let

$$\frac{T}{I} = b \text{ and } \frac{C}{I} = d \quad . \quad . \quad . \quad (7.130)$$

$$\int\frac{dt}{t} = -\int\frac{vdv}{v^2 - \frac{T}{I}v + \frac{C}{I}} = -\int\frac{vdv}{v^2 - bv + d} \quad . \quad (7.131)$$

$$= -\frac{1}{2}\int\frac{(2v - b)dv}{v^2 - bv + d} + \frac{b}{2}\int\frac{dv}{v^2 - bv + d} \quad . \quad (7.132)$$

$$\therefore \log_e t = -\frac{1}{2}\log(v^2 - bv + d) + \frac{b}{2}\int\frac{dv}{v^2 - bv + d} + k \quad (7.133)$$

The latter integral has different forms, depending on whether $b^2 \lessgtr 4d$, i.e. $\frac{T^2}{I^2} \lessgtr \frac{4C}{I}$.

Obviously $b^2 > 4d$, for

$$C = \frac{T\omega}{t} - \frac{I}{t}\frac{d\omega}{dt}\omega \quad . \quad (7.134)$$

$$\frac{C}{I} = \frac{T}{I}\frac{\omega}{t} - \frac{\omega}{t}\frac{d\omega}{dt} \quad . \quad . \quad . \quad (7.135)$$

$$\frac{T^2}{I^2} \gtrless 4\frac{T}{I}\frac{\omega}{t} - \frac{4\omega d\omega}{tdt} \quad . \quad . \quad . \quad (7.136)$$

but $I\frac{d\omega}{dt}$ usually $> T$

$$\therefore \qquad \frac{d\omega}{dt} > \frac{T}{I} = m\frac{T}{I} \qquad (7.137)$$

$$\therefore \qquad \frac{T^2}{I^2} + 4m\frac{T\omega}{It} > \frac{4T\omega}{It} \qquad (7.138)$$

$$\therefore \qquad \frac{T^2}{I^2} > \frac{4C}{I} \qquad (7.139)$$

$$\therefore \qquad \int \frac{dv}{v^2 - bv + d} = \int \frac{dv}{(v - \frac{1}{2}b)^2 - \left(\frac{b^2}{4} - d\right)} \qquad (7.140)$$

$$= -\frac{1}{\sqrt{\frac{b^2}{4} - d}} \coth^{-1} \frac{(v - \frac{1}{2}b)}{\sqrt{\frac{b^2}{4} - d}} \qquad (7.141)$$

$$\therefore \log_e t = -\tfrac{1}{2}\log(v^2 - bv + d) + \frac{b}{2\sqrt{\frac{b^2}{4} - d}} \coth^{-1} \frac{(v - \frac{1}{2}b)}{\sqrt{\frac{1}{4}b^2 - d}} + K \qquad (7.142)$$

When $t = t_1$, the time when the regulator comes in, $\omega = \omega_1$,

$$\therefore \; K = \log_e t_1 + \tfrac{1}{2}\log\left(\frac{\omega_1^2}{t_1^2} - \frac{b\omega_1}{t_1} + d\right) - \frac{b}{2\sqrt{\frac{b^2}{4} - d}} \coth^{-1} \frac{\left(\frac{\omega_1}{t_1} - \frac{b}{2}\right)}{\sqrt{\frac{1}{4}b^2 - d}} \qquad (7.143)$$

$$\therefore \; \log_e \frac{t}{t_1} = \frac{1}{2}\log\frac{\left(\frac{\omega_1^2}{t_1^2} - \frac{b\omega_1}{t_1} + d\right)}{\left(\frac{\omega^2}{t^2} - \frac{b\omega}{t} + d\right)} + \frac{b}{2\sqrt{\frac{b^2}{4} - d}}\left[\coth^{-1}\frac{\left(\frac{\omega}{t} - \frac{b}{2}\right)}{\sqrt{\frac{b^2}{4} - d}} - \coth^{-1}\frac{\left(\frac{\omega_1}{t_1} - \frac{b}{2}\right)}{\sqrt{\frac{b^2}{4} - d}}\right] \qquad (7.144)$$

We have, approximately (neglecting the second term)

$$\omega = \frac{\frac{T}{I}t \pm t\sqrt{\frac{T^2}{I^2} - 4\left\{\left(\frac{C}{I} - \frac{\omega_1^2}{t^2} + \frac{T}{I}\frac{\omega_1 t_1}{t^2} + \frac{t_1^2}{t^2}\frac{C}{I}\right)\right\}}}{2} \qquad (7.145)$$

This equation holds up to the peak. The output is constant from C to D and our equation becomes

$$T\omega - I\frac{d\omega}{dt}\omega = E \qquad . \qquad . \qquad . \quad (7.146)$$

where E = constant output

This equation is similar to equation (7.86), and is solved in the same way.

Speed Control by Pole-changing

Since the revolutions per minute $= \dfrac{120 \times \text{frequency}}{\text{number of poles}}$, it is clear that speed change can be obtained by changing the number of poles. This may be effected by a regrouping of the coils of a single winding, or by using two or more independent windings. For speed ranges of 2 : 1, this type of motor, using a single stator winding and a squirrel-cage rotor, is commonly used. There are a number of different ways of connecting the windings of such motors; the selection of the connection, in any case, depending on the relative maximum outputs required at the two speeds. The connection most frequently used, and in which the material is used to the best advantage, has a half-speed rating of from 60 to 70 per cent of the full-speed rating. The following table gives several different connections, which may be used, with the approximate relative outputs for the different connections—

	Speed	Connection	Approximate Maximum Output
(1)	100	2 circuit Δ	100
	50	Y-Δ	11
(2)	100	2 circuit Y	100
	50	1 circuit Y	22
(3)	100	2 circuit Y	100
	50	1 circuit Δ	66
(4)	100	1 circuit Δ	100
	50	2 circuit Y	117
(5)	100	1 circuit Y	100
	50	2 circuit Y	350
(6)	100	Y-Δ	100
	50	2 circuit Δ	700

The values are approximate only and will vary with the ratio of reactance to resistance, and also with the ratio of end connection reactance to the slot reactance. The method of pole changing is very simple. It is to reverse the current in half the winding for the smaller number of poles. The following diagram shows how it is done for four poles and two poles.

Fig. 7.5 shows a winding, for one phase, arranged for four poles. Fig. 7.6 shows the regrouping effected for two poles.

For the smaller number of poles it will be seen that one terminal is connected to the mid-point tapping A, and the other terminal

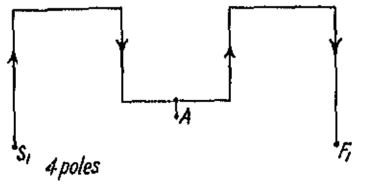


Fig. 7.5

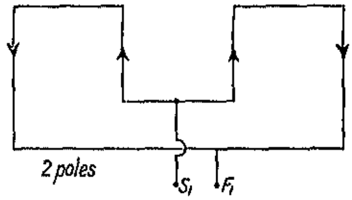


Fig. 7.6

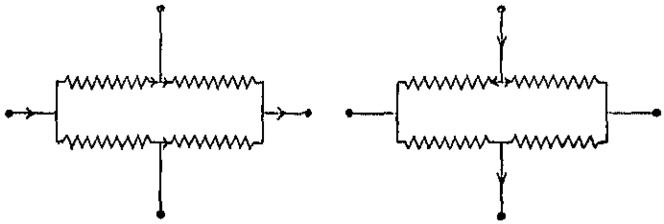


Fig. 7.7

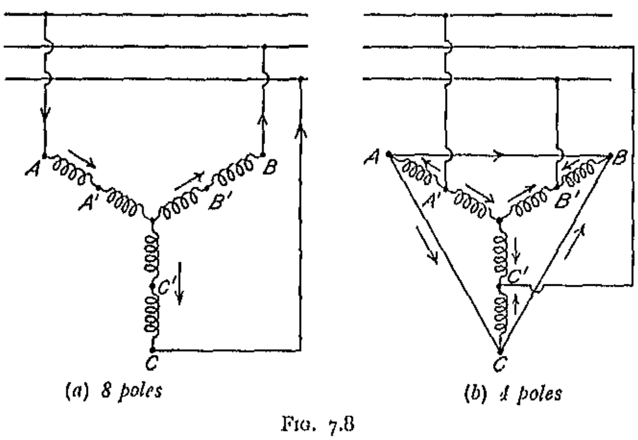


Fig. 7.8

of the winding is connected to the lead which joins the beginning and end of the winding together. If we wish to keep the same number of coils in series, in each case, the winding may be arranged as in Fig. 7.7.

In Fig. 7.8 (a) and (b) the winding is connected in star for the larger number of poles and in two parallel star for the smaller number of poles. In the case of a single-layer hemi-tropic winding for eight poles, each phase will consist of four groups of coils, two

groups straight and two groups bent, straight and bent coils alternating. Alternate coils are connected in series, i.e. two straight coils are connected in series and two bent coils are connected in series. For the eight-pole connection, the circuit is completed in series through the two bent coils of the same phase and a tapping is brought out at the junction of the straight and bent coils. A similar method is adopted in regard to the other phases. This method of connection is necessary in order that the direction of current in alternate groups of coils may be reversed when the two halves of each phase are connected in parallel.

In order that the direction of rotation shall remain unchanged when the poles are changed, two of the phases must be reversed in relation to the line wires, with this connection. The larger number of poles is obtained by connecting the line wires to the ends of the winding; while the smaller number is obtained by connecting the lines to the tappings, and short-circuiting the ends of the winding.

The development of a three-phase winding for pole-changing, according to Fig. 7.8 (*a*) and (*b*), is shown in Fig. 7.9.

It will be seen from Fig. 7.9 that, with the smaller number of poles, certain conductors, at the centre of each pole, carry currents in the wrong direction. A part of the winding is ineffective. For this class of winding one-third of the turns per pole are ineffective.

Let $T =$ turns per phase for the larger number of poles

$p =$ poles

For the smaller number of poles, i.e. $\frac{p}{2}$, the effective turns per phase will be

$$\frac{1}{2}\left(1\tfrac{2}{3}\,\frac{T}{p}\times\frac{p}{2}\right) = \frac{5}{12}\,T$$

Ratio of Fluxes and Magnetizing Currents

Let $E =$ counter e.m.f. per phase $\simeq V$

$\hat{\phi} =$ maximum flux per pole for larger number of poles

$\hat{\phi}' =$ maximum flux per pole for smaller number of poles

$f =$ frequency

$K_1 =$ breadth factor for p poles

$K_2 =$ breadth factor for $\frac{p}{2}$ poles

$K_3 =$ coil span factor for p poles

$K_4 =$ coil span factor for $\frac{p}{2}$ poles

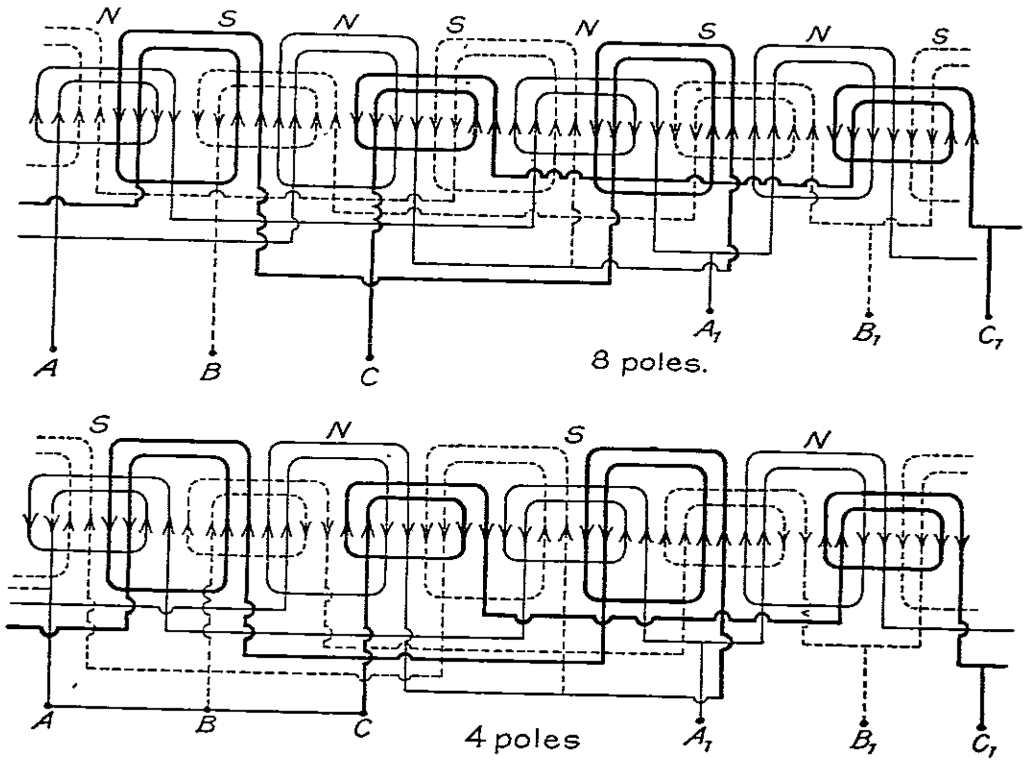


FIG. 7·9

Then for p poles

$$\hat{E}_1 = \sqrt{2}\pi \times K_1 \times K_3 \times \hat{\phi} \times T \times f \times 10^{-8}$$

for $\frac{p}{2}$ poles

$$\hat{E}_2 = \sqrt{2}\pi \times K_3 \times K_4 \times \hat{\phi}' \times \frac{5}{12} T \times f \times 10^{-8} \quad . \quad (7.147)$$

Let ψ = span of the coil, in electrical degrees; for the smaller number of poles $\psi = 90°$, corresponding to half-pitch

$$K_4 = \sin\frac{\psi}{2} = \sin 45° = 0{\cdot}707 \quad . \quad . \quad . \quad . \quad (7.148)$$

and since

$$\hat{E}_1 \simeq \hat{E}_2 \simeq V \quad . \quad . \quad . \quad (7.149)$$

$$\frac{\hat{\phi}'}{\hat{\phi}} = \frac{5}{12} \times 0{\cdot}707 = 3{\cdot}4 \quad . \quad . \quad . \quad (7.150)$$

The area of the pole face is doubled, so the flux density in the gap, with this connection, is 1·7 times that for the larger number of poles.

If one neglects saturation, the magnetizing current is proportional to the gap density, and inversely proportional to the turns per pole per phase.

$$\frac{i_m}{i_m'} = \frac{B_m}{\frac{T}{p}} \div \frac{B_m'}{0{\cdot}707 \times \frac{5}{12}\frac{T}{p} \times 2} \quad . \quad . \quad (7.151)$$

and

$$\frac{B_m}{B_m'} = \frac{1}{1{\cdot}7}$$

$$\therefore \quad \frac{i_m}{i_m'} = \frac{1}{1{\cdot}7} \times \frac{2 \times 5 \times 0{\cdot}707}{12} = 0{\cdot}347 \quad . \quad . \quad (7.152)$$

$$\therefore \quad i_m' = 2{\cdot}88 i_m \quad . \quad . \quad . \quad (7.153)$$

Careful attention must be paid with this method of connection, viz. star for larger number of poles and two parallel star for the smaller number, to the areas of core and teeth, if saturation is to be avoided on the smaller number of poles. This method of connection is *not* a good one and it is preferable to use—

(1) Delta connection for the larger number of poles.

(2) *Two parallel circuits per phase* in *star* for the smaller number of poles.

With (1) delta connection for p poles and (2) two parallel star for $\frac{p}{2}$ poles.

For the larger number of poles

(3) $$V \simeq E_1 = \sqrt{2}\pi\hat{\phi} \times T \times f \times K_1 \times K_3 \times 10^{-8} \quad . \quad (7.154)$$

(4) For $\frac{p}{2}$ poles

$$\frac{E_2}{\sqrt{3}} \simeq \frac{V}{\sqrt{3}} = \sqrt{2}\pi\hat{\phi}' \times \frac{5}{12} T \times 0{\cdot}707 \times f \times 10^{-8} \quad (7.155)$$

$$\therefore \qquad \frac{\hat{\phi}'}{\hat{\phi}} = \frac{12}{5 \times 0{\cdot}707 \times \sqrt{3}} = 1{\cdot}96 \quad . \quad . \quad (7.156)$$

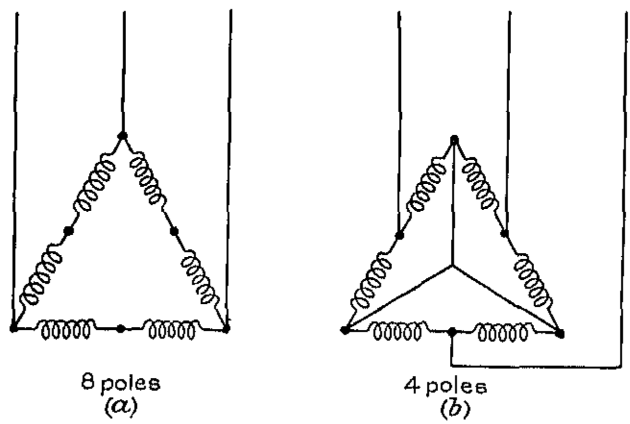


FIG. 7.10

Since the pole area is doubled,

$$B_m' = 0{\cdot}98B$$

and
$$\frac{i_m}{i_m'} = \frac{2 \times 0{\cdot}707 \times \frac{5}{12}}{0{\cdot}98} = 0{\cdot}6 \quad . \quad . \quad (7.157)$$

$$\therefore \qquad i_m' = 1{\cdot}66 i_m$$

This method of connection is preferable, since the performance is better and the motor can be made smaller in external diameter, and lighter than one connected as in Fig. 7.8. It may also be mentioned that, with the machine delta-connected for the larger number of poles and star-connected for the smaller number of poles, no reversal of line wires and phases is necessary. This connection is shown in Fig. 7.10 (*a*) and (*b*).

The wave form of m.m.f., on the smaller number of poles, is very irregular, giving rise to harmonics of large magnitude, especially when the coil pitch is chosen as full pitch for the larger number of poles. It will be shown, in the section on the revolving field, that in three-phase machines the fifth harmonic, the eleventh,

etc., travel in the opposite direction to the fundamental, while the seventh, thirteenth, etc., revolve in the same direction as the fundamental. These harmonic fluxes produce torques, by interaction with the currents in the rotor of corresponding frequency, and they modify, profoundly, the resultant torque-slip curve of the motor. The torque-slip curve is no longer smooth, but contains saddle-backs, and it may be that the motor may fail to accelerate to full speed. Indeed, the motor may run steadily at a sub-synchronous speed, and get very hot in the process. Furthermore, noise and vibration may be caused and, of course, excessive iron and eddy-current losses.

It should also be mentioned that, where separate windings are provided for the different pole numbers, the size of the motor is increased appreciably and, of course, the power factor falls greatly on the slow speeds.

For large speed-ranges, the control arrangements for changing the coil groups become complicated.

The problem of multi-speed motor design is so to arrange the windings that the change of connections, in changing from one number of poles to another, involves the least number of switching arrangements.

Speed-control by Cascade Connection

Assume that we have two induction motors, mechanically coupled together, and let the rotor winding of the first be connected to the stator winding of the second motor, and the rotor winding of the second either short-circuited or closed through external resistance. If the fields rotate in the same direction in both machines, the set will run at a speed corresponding to that of a motor having a number of poles equal to the sum of the numbers of poles in the two machines.

Let p_1 = number of poles in the first machine

p_2 = number of poles in the second machine

f = supply frequency

The synchronous speed of the first machine, in revolutions per second $= \dfrac{2f}{p_1}$

The synchronous speed of the second machine, when supplied at full frequency $= \dfrac{2f}{p_2}$ revolutions per second.

If s = slip of the first machine, then the speed of machine No. 1 is $(1 - s)\dfrac{2f}{p_1}$ revolutions per second.

The frequency of the rotor currents of machine No. 1 is sf, and this is the frequency of the stator currents of machine No. 2.

Let s' = the slip of motor No. 2

$$= \frac{\frac{2sf}{p_2} - (1 - s)\frac{2f}{p_1}}{\frac{2sf}{p_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.158)$$

$$s' = \frac{\frac{s}{p_2} - \frac{1-s}{p_1}}{\frac{s}{p_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.159)$$

The slip of the second motor, $s' = 0$,

when
$$\frac{s}{p_2} - \frac{1-s}{p_1} = 0 \quad . \quad . \quad . \quad (7.160)$$

$$\therefore \quad \frac{1-s}{s} = \frac{p_1}{p_2} \quad . \quad . \quad . \quad . \quad (7.161)$$

i.e.
$$\frac{1}{s} - 1 = \frac{p_1}{p_2} \quad . \quad . \quad . \quad (7.162)$$

i.e.
$$\frac{1}{s} = \frac{p_1 + p_2}{p_2} \quad . \quad . \quad . \quad (7.163)$$

$$\therefore \quad s = \frac{p_2}{p_1 + p_2} \quad . \quad . \quad (7.164)$$

The speed of the set, when $s' = 0$, in revolutions per second, i.e. the *cascade synchronous speed*,

$$= (1 - s)\frac{2f}{p_1}$$

$$= \left(1 - \frac{p_2}{p_1 + p_2}\right) \times \frac{2f}{p_1} \text{ revolutions per second}$$

$$= \frac{2f}{p_1 + p_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.165)$$

That is, the set runs at a cascade synchronous speed

$$\frac{2f}{p_1 + p_2} \text{ revolutions per second}$$

i.e. corresponding to a motor having a number of poles equal to the

sum of the number of poles in both machines. Thus, if $p_1 = p_2$ the set will run at half speed.

If the fields are arranged to rotate in *opposite* directions in both machines, then the speed of the set is that of a motor, having a number of poles equal to the difference of the numbers of poles in the two machines,

$$\text{i.e.} \qquad \frac{2f}{p_1 - p_2} \text{ revolutions per second} \qquad (7.166)$$

The Cascade System of induction motor control is shown in Fig. 7.11.

The method of building up the circle diagram for two motors in cascade is shown in Fig. 7.12.

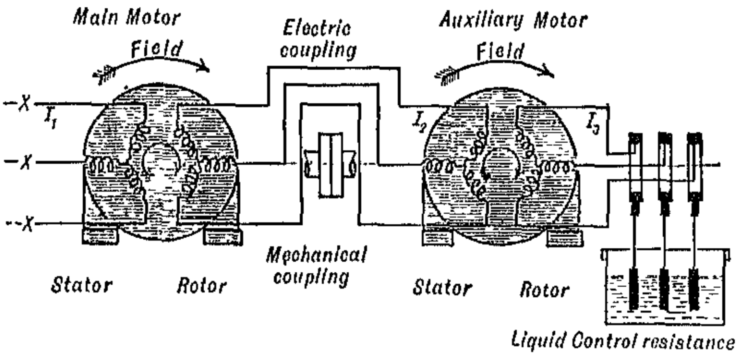


FIG. 7.11. CASCADE SYSTEM OF INDUCTION MOTOR CONTROL

The calculation of the performance of a cascade set is best effected by the aid of complex quantities. One starts with the current in the rotor of the auxiliary motor and works back to the primary of the first machine.

Let $\bar{E}$ = back e.m.f. generated in the second motor rotor by the flux linking both stator and rotor, *at full frequency*

x_4 = leakage reactance of the auxiliary rotor, at full frequency

$\bar{I}_1$ = rotor current of the auxiliary motor

s = slip of the first motor

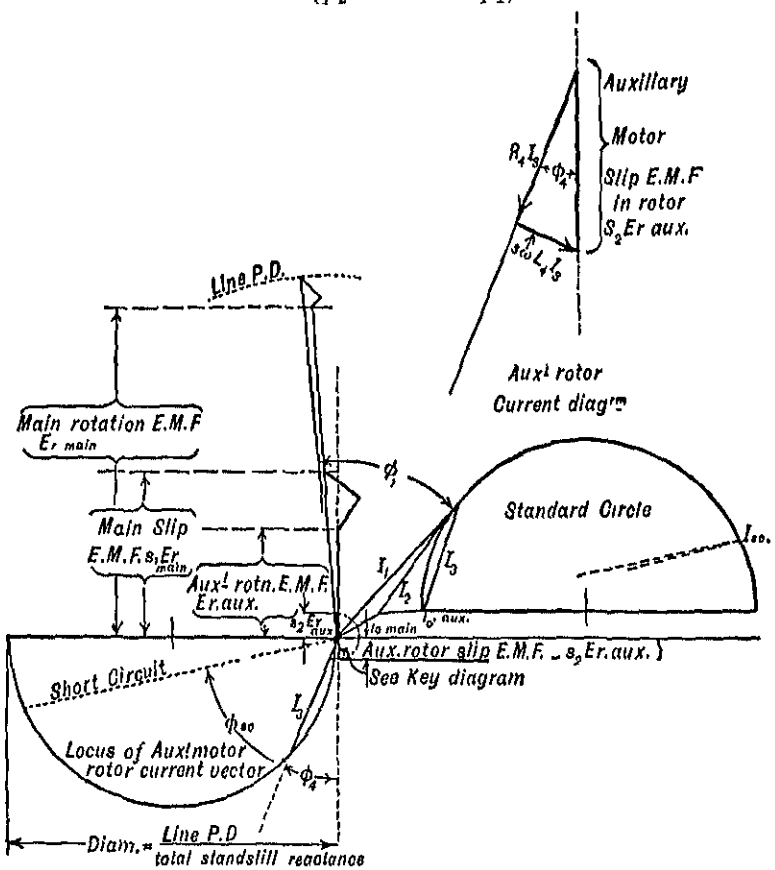
$a = \dfrac{p_2}{p_1}$ = ratio of number of poles

where p_2 = number of poles in motor No. 2

p_1 = number of poles in motor No. 1

The relative speed of rotation, with respect to the field in the auxiliary machine, in revolutions per second

$$= \left\{\frac{2fs}{p_2} - (1 - s)\,\frac{2f}{p_1}\right\}$$



PLANE I—Coils 1 and 4. PLANE II—Coils 2 and 5. PLANE III—Coils 3 and 6
Coils numbered round stator in clockwise direction
Start of coil to innermost slot coil of coil group
Finish of coil from outermost slot coil of coil group

FIG. 7.12. STANDARD CIRCLE DIAGRAM FOR TWO INDUCTION MOTORS IN CASCADE

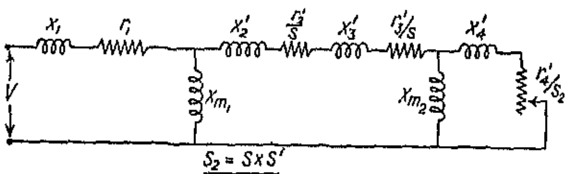


FIG 7.13. EQUIVALENT CIRCUIT FOR TWO INDUCTION MOTORS IN CASCADE

The frequency of the rotor currents in the second machine

$$= \left\{\frac{2fs}{p_2} - (1-s)\frac{2f}{p_1}\right\} \times \frac{p_2}{2}$$
$$= f\left\{s - (1-s)\frac{p_2}{p_1}\right\}$$
$$= f\{s - (1-s)a\} = f\{s(1+a) - a\}$$
$$= s_2 f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.167)$$

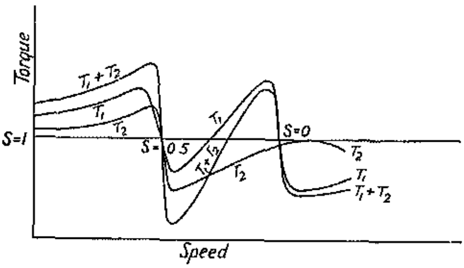


FIG. 7.14. TORQUE CURVES FOR TWO MACHINES IN CASCADE

$s_2 = s(1+a) - a.$

$$\text{Thus, } s_2 = s \times s' = s \times \frac{\left\{\dfrac{s}{p_2} - \dfrac{1-s}{p_1}\right\}}{\dfrac{s}{p_2}} = s(1+a) - a \quad . \quad (7.168)$$

s = slip of machine No. 1
s' = slip of machine No. 2

$$I_4 = \frac{s_2\mathbf{E}}{r_4 + js_2x_4} \quad . \quad . \quad . \quad . \quad (7.169)$$

= rotor current in motor No. 2

where r_4 = resistance per phase of rotor No. 2
x_4 = leakage reactance per phase of rotor No. 2 (at full frequency)
$\mathbf{E}$ = e.m.f. generated at standstill in rotor No. 2

$$I_4 = \frac{s_2r_4\mathbf{E}}{r_4^2 + s_2^2x_4^2} - \frac{js_2^2x_4\mathbf{E}}{r_4^2 + s_2^2x_4^2} \quad . \quad . \quad (7.170)$$

$$= \mathbf{E}(\alpha - j\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.171)$$

where $\alpha = \dfrac{s_2r_4}{r_4^2 + s_2^2x_4^2}$

$\beta = \dfrac{s_2^2x_4}{r_4^2 + s_2^2x_4^2}$

Now if $g_2 - jb_2$ = exciting admittance of the second motor. The no-load current

$$I_{\mu_2} = E'(g_2 - jb_2) \quad . \quad . \quad . \quad (7.172)$$

The primary current of motor No. 2

$$I_3 = \frac{I_4}{\alpha_2} + I_{\mu_2} \quad . \quad . \quad . \quad . \quad (7.173)$$

where α_2 = ratio of transformation of motor No. 2

and $E' = E \times \alpha_2$ . . . . (7.174)

Therefore, primary current of motor No. 2

$$I_3 = \frac{E(\alpha - j\beta)}{\alpha_2} + E'(g_2 - jb_2) \quad . \quad . \quad (7.175)$$

$$= \frac{E'}{\alpha_2^2}(\alpha - j\beta) + E'(g_2 - jb_2) \quad . \quad . \quad (7.176)$$

E' = e.m.f. generated, in stator No. 2, per phase at standstill,

$$\therefore \quad I_3 = E'(x - jy) \quad . \quad . \quad . \quad (7.177)$$

$$x = g_2 + \frac{\alpha}{\alpha_2^2} \text{ and } y = b_2 + \frac{\beta}{\alpha_2^2} \quad . \quad . \quad (7.178)$$

The voltage induced in the rotor of motor No. 1, at slip s

$$= sE_2 = sE' + I_3 Z \quad . \quad . \quad . \quad (7.179)$$

where E_2 = voltage generated in rotor No. 1, at standstill

and $Z = (r_2 + r_3) + js(x_2 + x_3)$ . . . . . (7.180)

where r_2 = resistance of rotor No. 1, per phase

r_3 = resistance of stator No. 2, per phase

x_3 = leakage reactance of stator No. 2, per phase, at full frequency

x_2 = leakage reactance of rotor No. 1, per phase, at full frequency

$$\therefore \quad E_2 = E' + \frac{I_3 Z}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.181)$$

$$= E' + \frac{E'(x - jy)\{(r_2 + r_3) + js(x_2 + x_3)\}}{s} \quad . \quad (7.182)$$

$$= E'(v + jw) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.183)$$

$$v = 1 + \frac{x(r_2 + r_3) + sy(x_2 + x_3)}{s} \quad . \quad . \quad (7.184)$$

$$w = x(x_2 + x_3) - \frac{y(r_2 + r_3)}{s}$$

Let $$\mathbf{E}_2' = \mathbf{E}_2 \times \alpha_1 \qquad (7.185)$$
where α_1 = ratio of transformation of motor No. 1

The no-load current of the first machine

$$= \mathbf{E}_2'(g_1 - jb_1) \qquad (7.186)$$
$$= \mathbf{E}'\{v + jw\}\{(g_1 - jb_1)\}\alpha_1$$
$$= \mathbf{E}'(p - jq) \qquad (7.187)$$

Now the primary current of the first machine

$$I_1 = \frac{I_3}{\alpha_1} + \mathbf{E}'(p - jq) \qquad (7.188)$$
$$= \frac{\mathbf{E}'}{\alpha_1}(x - jy) + \mathbf{E}'(p - jq) \qquad (7.189)$$
$$= \mathbf{E}'(m - jn) \qquad (7.190)$$

where $$m = \frac{x}{\alpha_1} + p \qquad (7.191)$$
$$n = \frac{y}{\alpha_1} + q \qquad (7.192)$$

The applied volts per phase of motor No. 1

$$V = \mathbf{E}_2' + ZI_1 \qquad (7.193)$$
$$= \alpha_1\mathbf{E}_2 + (r_1 + jx_1)\mathbf{E}'(m - jn) \qquad (7.194)$$
$$= \alpha_1\mathbf{E}'(v + jw) + \mathbf{E}'(r_1 + jx_1)(m - jn) \qquad (7.195)$$
$$= \mathbf{E}'(r + js) \qquad (7.196)$$

$$\therefore \qquad V = \mathbf{E}'(r + js) \qquad (7.197)$$

$$\therefore \qquad E' = \frac{V}{\sqrt{r^2 + s^2}} \qquad (7.198)$$

and $$I_1 = \frac{V\sqrt{m^2 + n^2}}{\sqrt{r^2 + s^2}} \qquad (7.199)$$

The power input to the set

$$= \text{real part of } VI_1$$
$$= \mathbf{E}'\{(r + js)\}\mathbf{E}'(m + jn) \qquad (7.200)$$
$$= (E')^2\{(rm - sn)\} \qquad (7.201)$$

power input to the set

$$= \frac{V^2}{r^2 + s^2}(rm - sn) \qquad (7.202)$$

Volt-ampere input $$= VI_1 \qquad (7.203)$$

Torque of the auxiliary motor

$$= E^2\alpha \qquad (7.204)$$

Torque of the main motor, real component

$$= E_2 I_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.205)$$

$$= E'(v + jw) E'(x + jy) \quad . \quad . \quad . \quad (7.206)$$

$$= (E')^2(vx - jw) \quad . \quad . \quad . \quad . \quad (7.207)$$

With two similar motors in cascade, the torque becomes zero at cascade synchronous speed. Above this speed, the torque of the second machine becomes a generator torque. Due to the reversal of the current in the secondary of the first motor, its torque becomes negative also. Above cascade synchronous speed the set acts as an induction generator. When approaching full synchronous speed, the generator torque of the auxiliary motor becomes very small and the second motor acts merely as an impedance in the secondary circuit of the first machine, which again exerts a motor torque. Thus, somewhere between half synchronous speed and full speed, the torque of the first motor becomes zero, while the second motor still gives a small generator torque. A little above this speed, the torque of the set becomes zero, and above this speed, the set gives a motor torque, although the auxiliary motor still gives a small negative torque. Above full synchronous speed, the set acts as generator but practically only the first machine contributes to the generator torque above, and the motor torque below full synchronous speed.

With high resistance in the rotor circuit of the auxiliary motor, the set gives a generator torque above half synchronous speed and remains a generator torque at all higher speeds. Instead of connecting the rotor windings of the first machine to the stator of the second we may connect the two rotor windings in series, and insert the resistance for starting or speed control, in the stator of the second machine, and avoid the use of slip-rings. The phases of the second rotor must be so connected to give a direction of rotation of the auxiliary field opposite to the rotation of the set. With two machines in cascade, the first motor carries the magnetizing current of both machines. The total self-inductive impedance of the two machines is the sum of the separate impedances. It is clear, therefore, that the dispersion coefficient of the set is practically four times that of one machine (assuming two similar machines in cascade), and, therefore, the power factor is rather low, even when each machine has a small number of poles. The output of the set, when running in cascade, is equal approximately to that of one machine at full speed.

Cascade connection is used on the Italian railways.

Combined with pole-changing it gives a variety of speeds and, of course, regenerative braking can be used, which is a very useful feature. By the combination of pole-changing and cascade connection, one can get a range of speeds efficiently, but there are wide gaps

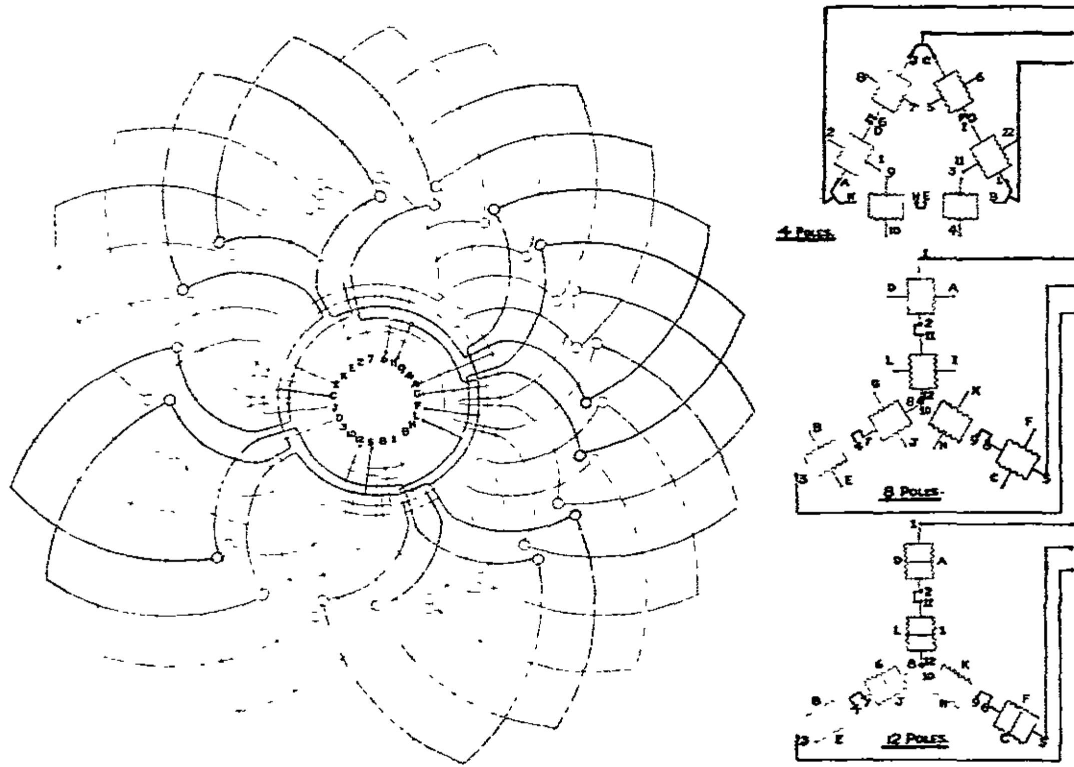


Fig. 7.15 Stator Winding suitable for Four, Eight or Twelve Poles

in the speed-range and continuous speed changes, such as are obtainable in the shunt d.c. motor by field control, are only obtainable by other means. Attempts were made to combine the two cascade machines into one and Lydall patented such a motor in 1902, in which the two windings were arranged in the same slots in stator and in rotor. The two sets of windings were wound for dissimilar numbers of poles, so that the stator windings were mutually non-inductive. The need to use deep slots to accommodate both windings caused a high leakage reactance and poor power factor, efficiency, and overload capacity.

Hunt went further and substituted one winding to perform the function of two separate windings. The stator winding must be suitable for the circulation of two independent currents, namely the main current of the frequency of supply and also the induced currents of slip frequency. The connections were such that these slip-frequency currents could only circulate when paths outside the winding were provided for them. It will be seen that control of starting torque and speed is effected by the use of resistance, connected to the stator circuits. Thus, the use of slip-rings was avoided in some cases. Mr. Hunt, at the time of his invention, was chief engineer of an important company, whose activities lay chiefly in the manufacture of mining machinery, and it will be appreciated that the elimination of possible sparking at slip-rings was a very desirable feature in machines operating in mines. Two sets of terminals were required, one set for connecting to the mains and the other set to the resistances. The main currents must not flow through the external secondary paths and, therefore, it is essential to connect the secondary terminals to two points, in the stator winding, between which no "main" potential difference exists. A "parallel-connected" winding is necessary for the stator.

There are two conditions to be satisfied, namely –

(1) The numbers of poles in the two fields must be so chosen that their windings are mutually non-inductive.

(2) The two fields, when superimposed, must not produce an unbalanced radial pull on the rotor.

These conditions are satisfied when the two numbers of poles are such that when divided by their greatest common factor the quotient is in one case even and in the other odd. Further, the greatest common factor must be greater than two. These rules are not perfectly general, as has been shown by Creedy. Any two numbers of poles must be even and must, therefore, have a common factor of 2. Hence their highest common factor must be a multiple of 2. This means that the two numbers of *pairs* of poles must have a common factor. The circumference of the machine is divided up into two or more identical sections, so that any values of magnetic density, which occur at any point, also occur at a point diametrically opposite, or at other equally spaced points, according as the common factor is

2, 3, 4, etc. The rule for magnetic balance is: the two pole numbers must differ by more than two to secure magnetic balance.

Mr. Creedy describes types of stator windings in which it is not necessary to have any common factor other than unity between the

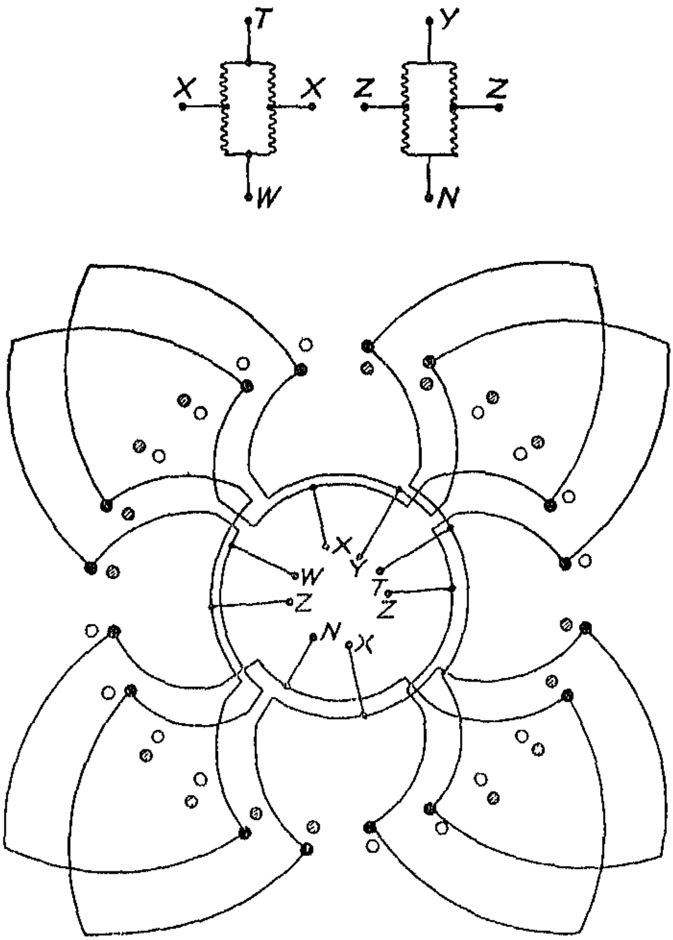


Fig. 7.16

two numbers of pairs of poles, in which case both numbers of pairs of poles must be odd. This gives machines of higher speed than is possible with the Hunt rule.

Fig. 7.15 shows the stator winding for four, eight, and twelve poles. The figures and letters, on the key diagram, correspond to those on the winding diagram.

Fig. 7.16 shows one phase of a pole-changing winding for the stator, suitable for four or eight poles.

Fig. 7.17 shows the windings of one phase of an eight-pole stator, with, for simplification, one slot per pole per phase. Each radial pair of small circles represents the two coil sections, occupying one slot, and the crosses and dots indicate the directions of the main currents. The main currents produce eight poles and the coils must be so connected that currents, induced by a four-pole field, can circulate in them. This four-pole field is shown in Fig. 7.17.

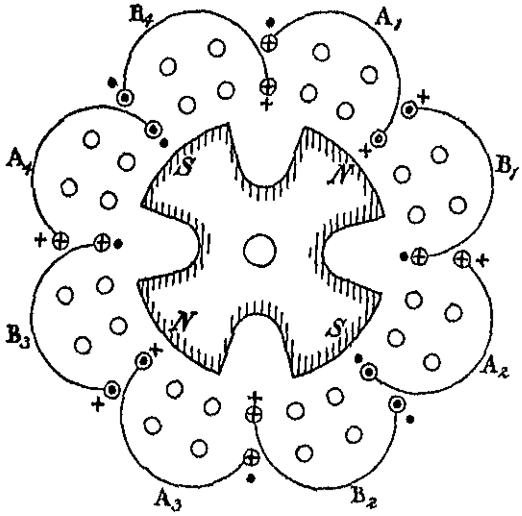


FIG. 7.17

Four of the slots are opposite the pole centres and four are situated midway between them.

The e.m.f.s induced in the conductors at the centres of the poles will be in quadrature with those induced in the conductors midway between the poles. The coils are marked A and B alternately, and all the induced currents in the A coils are in phase with one another, and in quadrature with those in the B coils. Fig. 7.18 shows the key diagram and Fig. 7.19 shows one phase connected up to comply with this diagram.

The main currents in the key diagram are shown by arrows inside the windings; the induced currents are shown by the arrows outside. All the A coils are included between T and C, and the B coils between C and the star point.

The rotor winding is exceedingly ingenious. The ratio of the numbers of turns in the main and auxiliary windings of the rotor is 1·73 to 1·0. This gives an auxiliary field with a flux per pole 73 per cent greater than that of the main field and a gap density of 86·5 per cent of that of the main field. The auxiliary magnetizing current is 75 per cent of the main-field magnetizing current. With a given

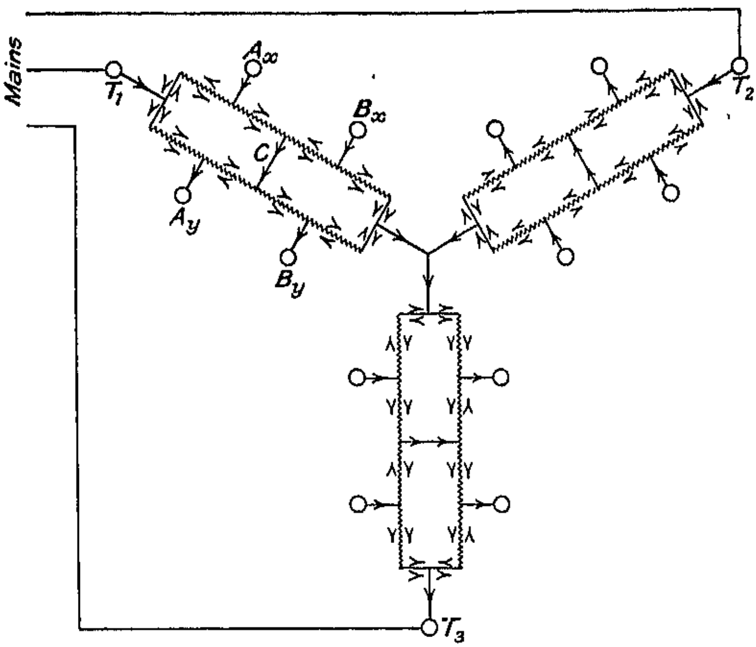


FIG. 7.18

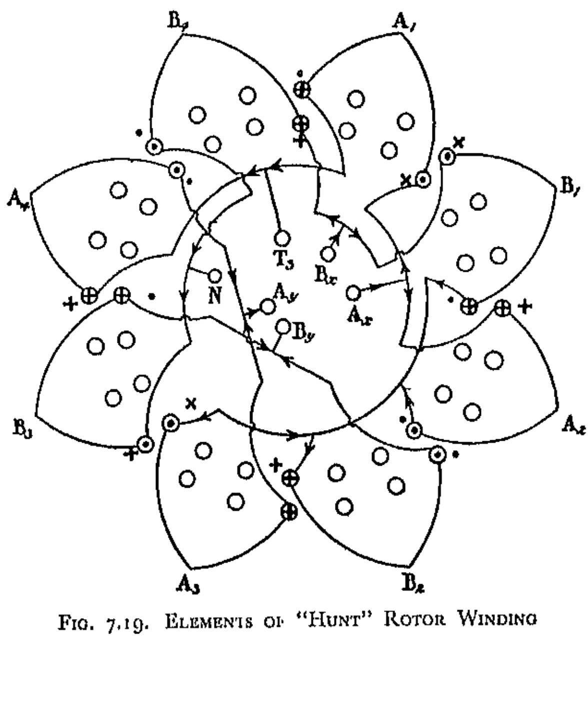


FIG. 7.19. ELEMENTS OF "HUNT" ROTOR WINDING

stator winding the magnetizing current of the auxiliary field varies inversely as the square of the number of turns in the auxiliary winding. The rotor winding consists of a main rotor winding, connected in star, and an auxiliary rotor winding connected in mesh.

Fig. 7.20 shows the elements of the two windings and the method of connection.

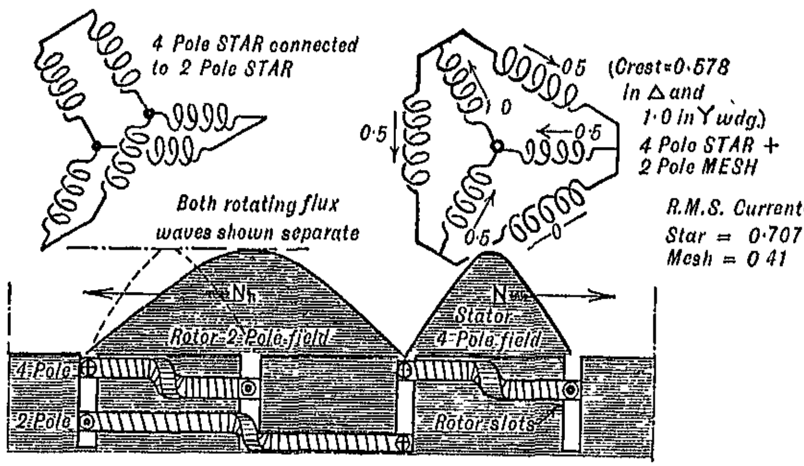
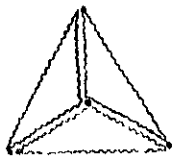


FIG. 7.20. CASCADE ROTOR WINDING

Fig. 7.21 shows a cascade rotor winding for twelve poles. The main and auxiliary rotor windings are superimposed to form a single winding.

In order to show how this superposition and combination is effected, a four-pole and two-pole combination is shown in Fig. 7.22. This is merely for the purpose of illustration and could not be used in practice, but it serves as a unit from which to build up other pole combinations.

The upper diagram of Fig. 7.22 shows the component windings for a four-pole and two-pole rotor, wound in twelve slots. It will be noticed that, in some slots, the currents are equal and flow in opposite directions and hence neutralize each other. These conductors are omitted and the remaining conductors are then connected up to form a resultant winding, which produces the four-pole and two-pole fields. Six slots have three bars and six have one bar per slot. The three bars are combined into two, so that one carries a double current and the other a single current. In one set of bars the currents are $\sqrt{3}$ times the currents in the other bars. These bars, carrying the $\sqrt{3}$ times the current in the other bars, are then connected in star and supply the single current bars, which are connected in delta. This arrangement saves 25 per cent of copper and reduces the inductance.



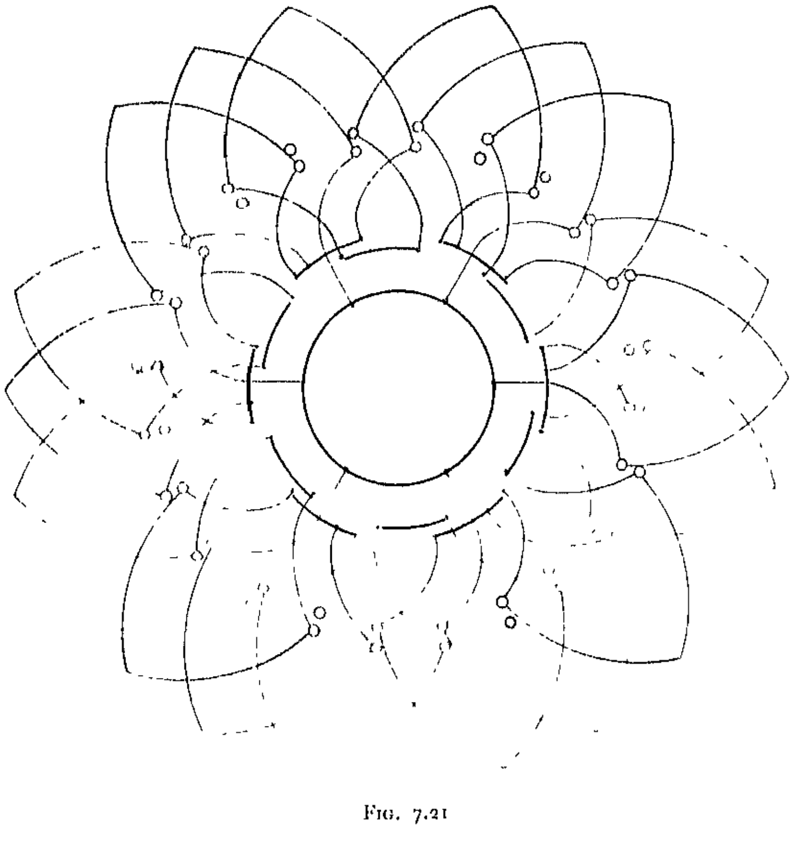


Fig. 7.21

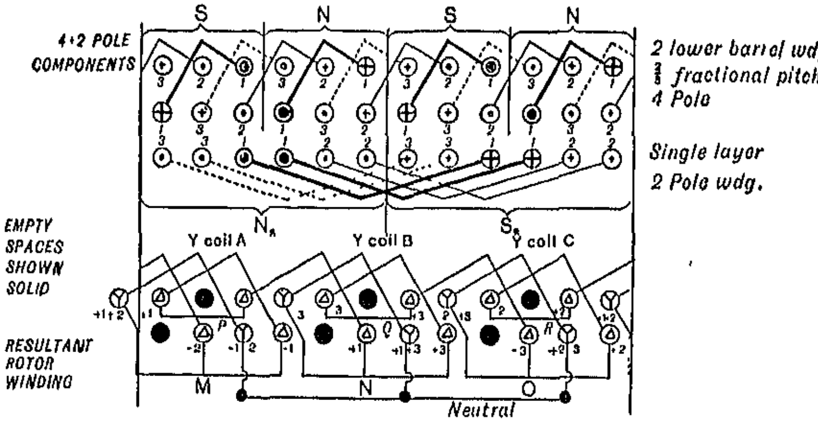


FIG. 7.22. DIAGRAMS OF Y. Δ ROTOR RESULTANT WINDING

Fig. 7.23 shows the resultant diagram.

The e.m.f.s in the mesh-connected winding, due to the main field, are balanced by the e.m.f.s generated by the auxiliary field, and current can circulate only by way of the star.

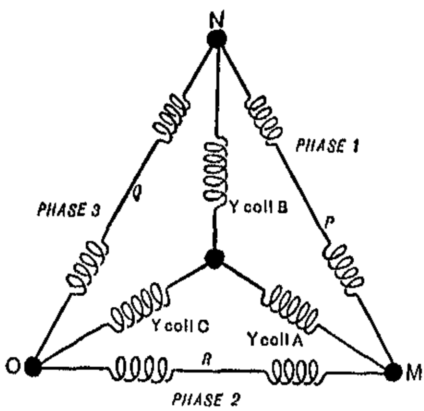


FIG. 7.23

Speeds, other than cascade speed, are obtained by bringing the ends of the mesh to slip-rings, which can be short-circuited. If the mesh windings are short-circuited, the motor will run as an induction motor having the main number of poles. The two-speed motor can be operated at a third speed, if the stator windings are so arranged that they can produce two different numbers of poles.

Fig. 7.24 shows an eight- + four-pole stator winding, with two groups of coils in each phase, divided and the neutral point opened.

To connect this winding for eight poles, two groups of coils in adjacent sections are placed in series and then, by means of a suitable switch, the whole winding is connected in star. (*See* Fig. 7.25.) Fig. 7.26 shows the same winding reconnected to give four poles.

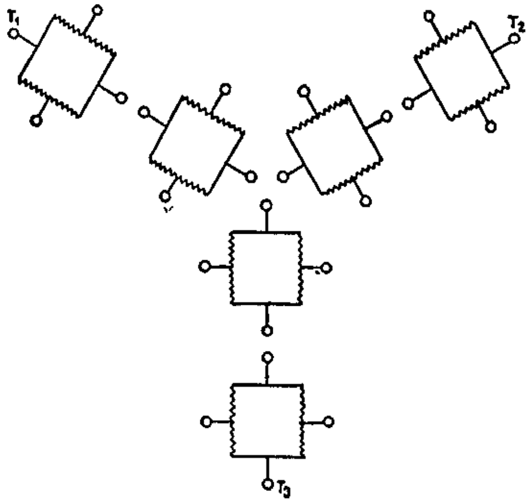


FIG. 7.24

Fig. 7.27 shows a cascade rotor winding suitable for four, eight, or twelve poles (twelve poles cascade).

It should be noted that the cascade machine is essentially a low-speed machine and, as pointed out by Mr. Hunt, shows to best advantage when designed for speeds which are abnormal for single-field motors.

There is a trend to-day, in marine work, to use alternating current on board ship, and the question of variable-speed motors for winches, etc., is a pressing one. The solution for many of these problems may be found in the use of a Hunt motor, combined with resistance in the secondary circuits. Another solution is the use of an induction motor driving McFarlane converters, each converter supplying two winches.

Change of Speed by Varying the Supply Frequency

Since r.p.m. $= \dfrac{120 \times f}{\text{poles}}$, it is clear that the speed may be varied by changing the supply frequency. The generators, driven by diesel engines or steam turbines, may have their speeds adjusted over a fair range, and the induction motors, supplied from the generators, will vary their speed as the frequency is varied. This is done usually on ships having an electric drive. On warships, where cruising and

fighting speeds are required, a low-frequency generator is provided for the low-speed, and separate and more powerful generators of higher frequency, for the high speed. If it is desired to use all the

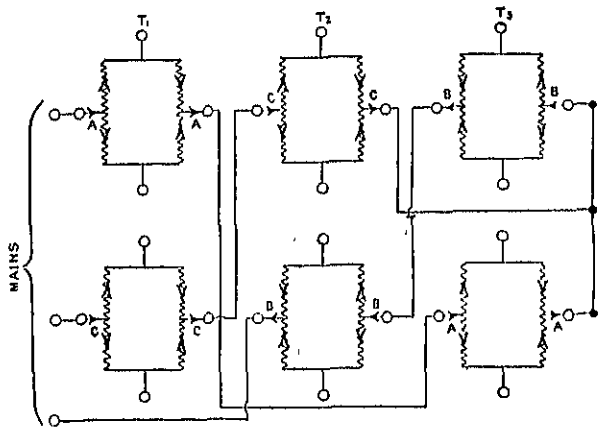


Fig. 7.25

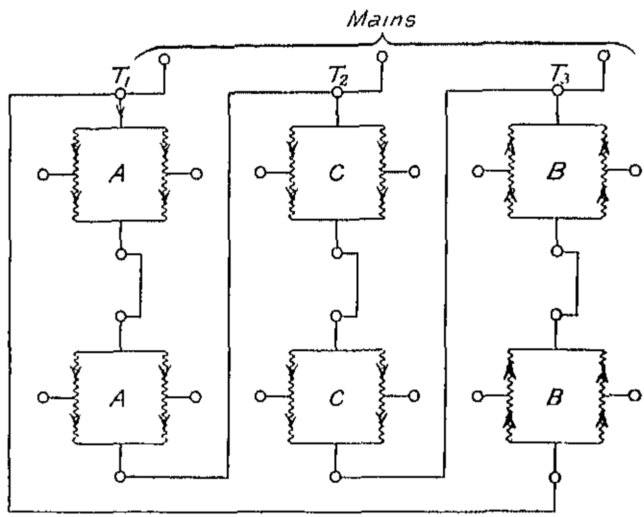


Fig. 7.26. Cascade Stator Winding Suitable for Four Poles

generators at the high speed, then all that is necessary is to use motors giving the higher speed at the low frequency.

The Krämer System of Speed Control

In this system the rotor winding of the induction motor is connected to the slip-rings of a rotary converter. The slip energy is converted

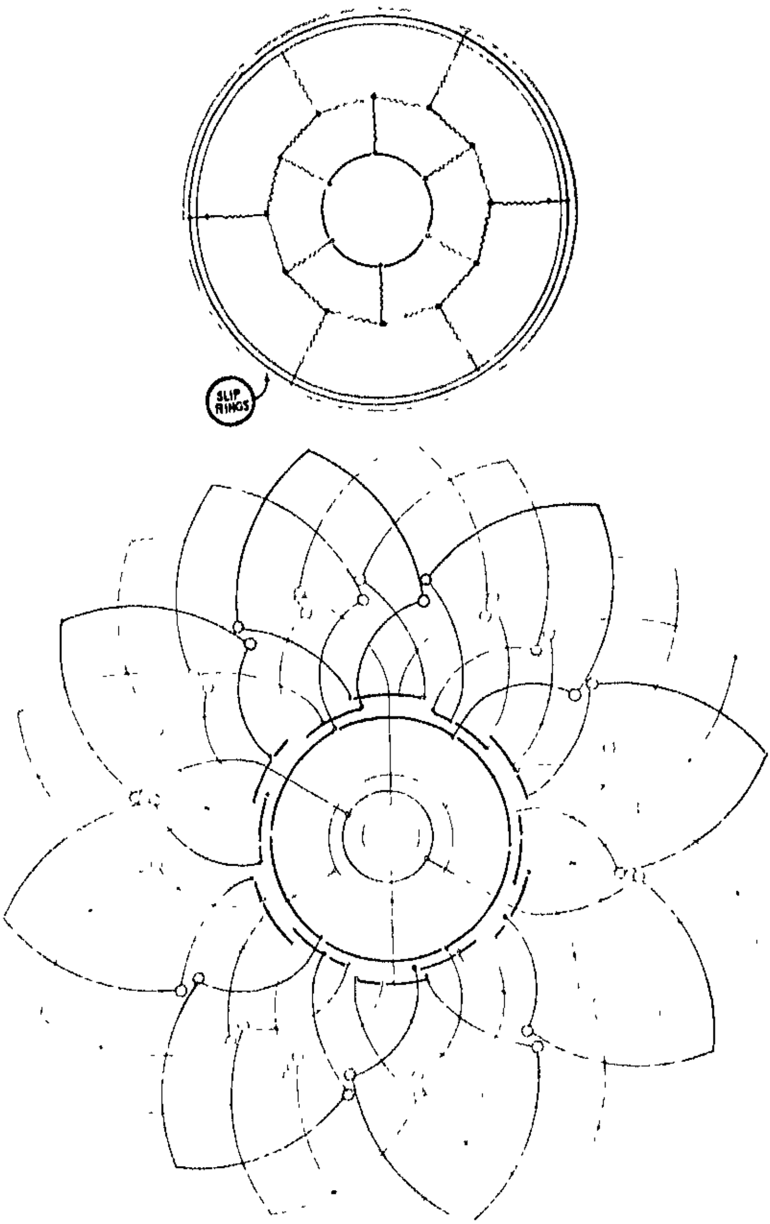


FIG. 7.27

into continuous-current energy, which can be given to a continuous-current network, or to a continuous-current auxiliary motor, coupled to the shaft of the induction motor. As the slip increases the power given out by the auxiliary motor rises by the same amount as that given out by the main motor falls. Thus, the sum of the outputs of the two motors remains constant over the whole-speed range, i.e. the torque varies inversely as the speed. This is a property frequently required in rolling-mill work. When working in conjunction with a flywheel, the requisite drop in speed can be obtained by a compound winding on the auxiliary d.c. motor. The auxiliary motor determines the speed of the set. To lower the speed, the field of the d.c. motor must be increased.

With the connections shown in Fig. 7.28 no special synchronizing gear is necessary. On switching on the excitation of the auxiliary

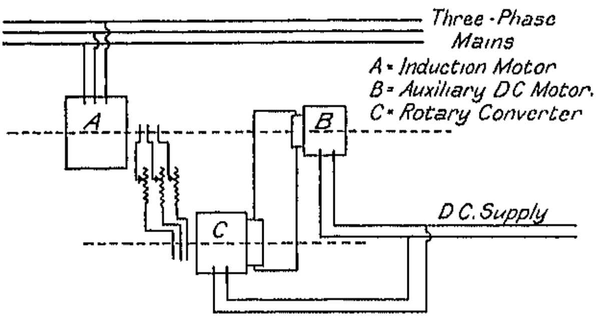


FIG. 7.28

motor and converter, the latter runs up to speed and falls into step. The maximum speed of the set is determined by the lowest frequency necessary for the stable running of the rotary converter. This is about 2 or 3 c/s, so that with a frequency of 50 c/s the highest speed will be about 4 to 6 per cent below synchronous speed of the main motor, or, if artificially raised above synchronism, 4 to 6 per cent above this will be the lowest stable speed. If the voltage on the slip-rings becomes small, as compared with the ohmic drop, the converter becomes unstable. The power factor of the set can be made unity over nearly the whole range by adjusting the excitation of the converter. If the speed of the induction motor is very low, the cost of a d.c. motor, on the same shaft, may be excessive. It is preferable, in this case, to return the slip energy to the supply system, by means of a high-speed motor-generator set. The generator may be of the asynchronous type. For work which requires constant torque the latter method is suitable.

The Krämer system is shown above in Fig. 7.28.

It will be seen that this method of controlling speed is expensive, in that costly auxiliary machines are required.

VARIATION OF SPEED AND POWER FACTOR BY THE USE OF THREE-PHASE COMMUTATOR MOTORS

Instead of converting to continuous current the slip-energy may be supplied to a three-phase commutator motor. This three-phase auxiliary motor is then coupled direct or by means of a belt-drive to the main motor. The diagram of connections is given in Fig. 7.29.

According to the characteristic required, a series motor, with brush-shifting device, or a shunt or compound motor with voltage regulation is used as auxiliary motor. With this arrangement the

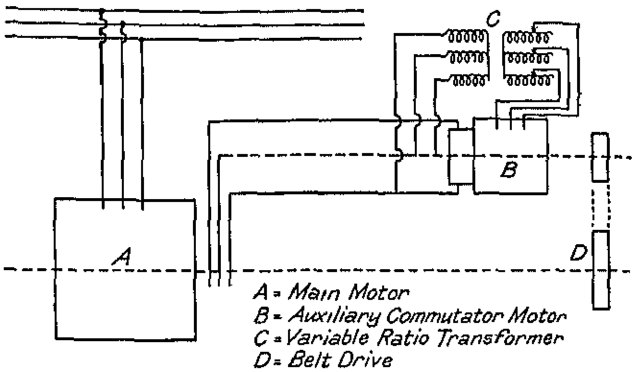


FIG. 7.29. NEUTRALIZED SERIES-EXCITED THREE-PHASE A.C COMMUTATOR MOTOR AND CONNECTIONS

output is constant for a given input, so that the torque varies inversely as the speed.

The Scherbius System

In this system, used by Brown-Boveri, the rotor current is led to a three-phase commutator motor. The latter is not coupled to the main motor but to a three-phase generator, which returns the slip energy to the line. The generator is of the induction type, running above synchronous speed. The speed characteristic obtained depends on the type of commutator motor, i.e. whether series, shunt, or compound.

This system is chiefly used for drives where the power falls with the speed, as in centrifugal machines. It has the advantage that the main motor can be run independently of the auxiliary set. The most common arrangements of regulating sets are (1) a series commutator machine for single-range regulation, i.e. for speed ranges below synchronism; (2) a shunt-wound commutator machine for single-range regulation; and (3) a shunt-wound commutator machine

with special exciter, for double-range speed regulation above and below synchronism.

The commutator machine is mechanically coupled to an induction machine and, in the case of single-range regulation below synchronism, the induction machine acts as generator. In double-range sets, the induction machine will act as a generator below synchronism, and as a motor above synchronism. The speed of the regulating set will change but little with change of speed of the main motor, since a very small percentage change of speed is sufficient to change from full-load motor to full-load generator action. It may be regarded, therefore, as having an approximately constant value. In all these cases an e.m.f. is injected into the rotor circuit of each phase. If this injected e.m.f. has a component, in phase opposition to the actual e.m.f. producing the rotor current, i.e. the e.m.f. overcoming the resistance drop, and tending to reduce the rotor current, it is clear the slip of the induction motor must increase to such a value that the slip e.m.f. will be sufficient to overcome the injected e.m.f. plus the resistance drop, due to the current required for the load torque.

Likewise, by adjusting the phase of the injected e.m.f. we can control the power factor.

Speed and power-factor control is, therefore, obtained by injecting an e.m.f. into the rotor circuit of the induction motor. If this e.m.f. opposes the slip e.m.f. the speed must fall; if it assists the speed must rise, because now a smaller slip is required to produce the rotor current, since the injected e.m.f. assists the slip e.m.f.

Clearly, also, if the injected e.m.f. has a component in quadrature with the slip e.m.f. of the main motor, the current will either lag or lead this e.m.f., depending on the phase of the injected e.m.f.

We shall be making a study of the three-phase commutator motor later, but it can be seen that the injected e.m.f., in the series machine, is proportional (neglecting saturation) to the rotor current of the main motor, since the speed of the commutator motor is practically constant and its phase is controlled by brush position.

Speed Regulation Below Synchronism

The diagram of connections is shown in Fig. 7.30. The form of circle diagram used, in this analysis, is one used by Behrend and Blondel, and considers only resistance drops and *not* leakage reactance voltages. These leakage reactance voltages are represented by corresponding *leakage* fluxes. The fluxes produced by primary and secondary currents are divided into fluxes which (*a*) link one winding (total flux); (*b*) mutual flux; and (*c*) leakage flux.

Fig. 7.31 gives a very clear picture of the performance of an induction motor working in conjunction with commutator sets.

Dr. Meyer-Delius and Mr. John I. Hull gave this analysis.

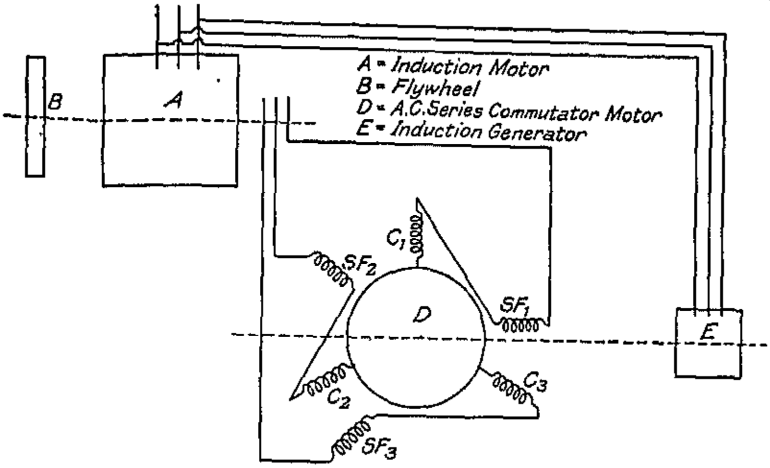


FIG. 7.30

For automatic single-range regulation of an induction motor equipped with a flywheel, to reduce peaks on line. SF_1, SF_2, and SF_3 are the series coils, C_1, C_2, and C_3 are neutralizing windings, neutralizing the armature ampere-turns

In Fig. 7.31

$AH = I_1$, proportional to the primary current, and represents the flux linking *the primary and secondary*, which would be produced by the primary current alone, neglecting saturation

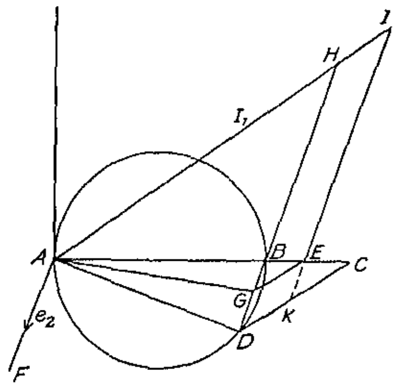


FIG 7.31

$HI = C_1I_1$ and represents primary leakage flux

$HG = IE = I_2$, proportional to the secondary current, and represents the flux linking both primary and secondary, which would be produced by the secondary current alone, neglecting saturation

$GD = C_2I_2$, represents secondary leakage flux

AE represents the total flux linking the primary, and represents the vector sum of the fictitious total primary flux and the fictitious flux, produced by the secondary current, which links the primary.

Neglecting resistance drop, E is a fixed point for constant line volts and frequency, since AE is the flux which generates the counter e.m.f. to balance the applied volts. HG intersects AE at B, and it is clear that

$$\frac{AB}{AE} = \frac{AH}{AI} = \frac{I_1}{I_1(1 + C_1)} = \frac{1}{1 + C_1} \quad . \quad . \quad (7.208)$$

Therefore, $AB = \dfrac{AE}{1 + C_1}$ and, since AE is constant, B is a fixed point.

AD is the resultant of AH and DH, i.e. of I_1 and $I_2 + C_2I_2$, and generates all secondary e.m.f.s, except the resistance drop, which is in phase opposition to e_2, set up in the secondary, in quadrature with the flux AD.

AF is, therefore, parallel to HD and to IE, and the angle ADH is a right angle. A line, parallel to AI from D, intersects AE produced in C.

$$BD = HD - HB = I_2(1 + C_2) - \frac{I_2}{1 + C_1} \quad . \quad (7.209)$$

since

$$\frac{HB}{I_2} = \frac{I_1}{I_1(1 + C_1)} \quad . \quad . \quad . \quad (7.210)$$

$$\therefore \quad BD = \frac{I_2}{1 + C_1}\{C_1 + C_2(1 + C_1)\} \quad . \quad . \quad (7.211)$$

$$\frac{CD}{AI} = \frac{BD}{IE} \quad . \quad . \quad . \quad . \quad (7.212)$$

$$\therefore \quad CD = I_1(1 + C_1) \times \frac{I_2\{C_1 + C_2(1 + C_1)\}}{I_2(1 + C_1)}$$

$$= I_1[C_1 + C_2(1 + C_1)] \quad . \quad . \quad . \quad . \quad (7.213)$$

Now EA represents the flux, whose counter e.m.f. balances the applied voltage. Let EA be represented by I_m.

Then

$$BA = I_0 = \frac{I_m}{1 + C_1} \quad . \quad . \quad . \quad (7.214)$$

$$\frac{CB}{CD} = \frac{AE}{AI} = \frac{I_m}{I_1(1 + C_1)} \quad . \quad . \quad . \quad (7.215)$$

$$\therefore \quad CB = CD \times \frac{I_m}{I_1(1 + C_1)} \quad . \quad . \quad . \quad (7.216)$$

$$= \frac{I_m[C_1 + C_2(1 + C_1)]}{1 + C_1} \quad . \quad . \quad (7.217)$$

$$= I_0[C_1 + C_2(1 + C_1)]. \quad . \quad . \quad (7.218)$$

$$CA = EA + CE = I_m(1 + C_2) \quad . \quad . \quad (7.219)$$

since
$$\frac{CE}{EK} = \frac{I_m}{I_2}$$

$$\therefore \qquad CE = I_m \times \frac{C_2 I_2}{I_2} = C_2 I_m \quad . \quad . \quad . \quad (7.220)$$

$$CA = I_m(1 + C_2) = \frac{I_m(1 + C_2)}{C_1 + C_2(1 + C_1)} \times [C_1 + C_2(1 + C_1)] \quad (7.221)$$

CA is, therefore, constant for constant I_m, and C is a fixed point. By selecting a suitable scale, I_m could be made to represent the magnetizing current for the whole flux, which is the usually calculated quantity. I_0 could be made to represent the running-light current. By changing the scale of the diagram by the factor $C_1 + C_2(1 + C_1)$, we may say that $CB = \frac{I_m}{1 + C_1} = I_0$; the primary current $I_1 = CD$

and
$$\frac{I_2}{1 + C_1} = BD \quad . \quad . \quad . \quad (7.222)$$

At standstill, with zero secondary resistance, AD, the resultant secondary flux must be zero, which means that D coincides with A and $CD = CA$, i.e. the ideal short-circuit current at standstill $= CA$. Since $C_1 I_1$ is defined as the primary leakage flux, the primary leakage reactance drop, with current I_m is $C_1 E_1$, since I_m produces the flux, which generates E_1.

If X_1 = primary leakage reactance

$$C_1 E_1 = I_m X_1 \quad . \quad . \quad (7.223)$$

$$\therefore \qquad C_1 = \frac{I_m X_1}{E_1} \text{ and } C_2 = \frac{I_m X_2}{E_1} \quad . \quad . \quad (7.244)$$

In order to draw the diagram, we need to know the primary and secondary leakage reactances X_1 and X_2, and the magnetizing current I_m.

The diagram is shown in Fig. 7.32.

In Fig. 7.32
$$CB = I_0 \quad . \quad . \quad . \quad . \quad (7.225)$$

$$CA = \frac{I_m(1 + C_2)}{C_1 + C_2(1 + C_1)} \quad . \quad . \quad . \quad (7.226)$$

CM = watt component of the input current

$$CD = I_1$$

The secondary current is $BD(1 + C_1) = I_2$.

To find the second voltage $\bar{E}_2 = AF$, one can first find its value at full frequency. Since AE (Fig. 7.31) is proportional to $\bar{E}_1$ at full frequency

$$AB = \frac{\bar{E}_1}{1 + C_1}$$

also

$$AD = \frac{AD}{AB} \times \frac{\bar{E}_1}{(1 + C_1)} \quad . \quad . \quad . \quad (7.227)$$

at full frequency. The actual value of the secondary volts $= I_2 r_2$, so the percentage slip

$$= \frac{I_2 r_2}{AD} \quad . \quad . \quad . \quad . \quad . \quad (7.228)$$

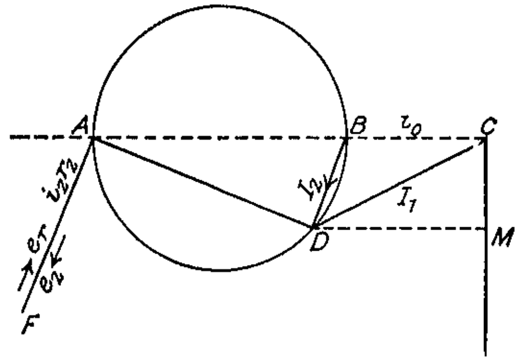


FIG. 7.32

The torque per phase, in synchronous watts,

$$= BD \times AD \quad . \quad (7.229)$$

The output per phase

$$= (1 - s)BD \times AD \quad . \quad (7.230)$$

The efficiency

$$= \frac{(1 - s)BD \times AD}{\bar{E}_1 \times MC} \quad . \quad . \quad (7.231)$$

and

$$\cos \phi = \frac{MC}{CD} \quad . \quad . \quad . \quad . \quad (7.232)$$

Now it is clear, from this simple diagram, that if an injected $\bar{E}_r'$, due to the commutator motor, exists in the rotor and if this e.m.f. is in phase opposition to the secondary current, the slip of the motor must increase to make the generated e.m.f. $\bar{E}_2 = \bar{E}_r + I_2 r_2$. In this case the slip is increased, for the effect of the injected e.m.f. is equivalent to an addition of resistance in the rotor circuit. If $\bar{E}_r$ is in phase with $\bar{E}_2$, and assists $\bar{E}_2$, the slip will decrease.

Now it is possible to inject, by means of the commutator motors, e.m.f.s having any phase angle with the rotor current. If the injected e.m.f. has a component, which is 90° ahead of BD, the effect will be to make BD lead. In other words, speed control and power factor correction are possible.

With the neutralized, series, commutator motor, shown in Fig. 7.30, the e.m.f.s at the terminals of the commutator motor are the leakage reactance drop, the resistance drop, and the rotational voltage of the armature. This rotational e.m.f. is proportional to the flux and speed of rotation, and the flux is proportional to the secondary

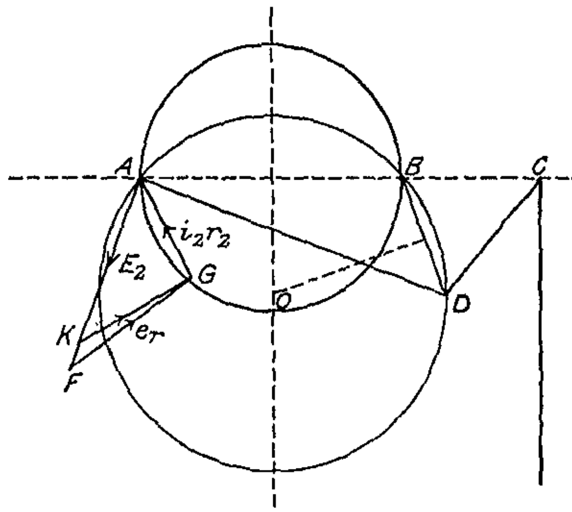


FIG. 7.33

currents. Since the speed of the commutator motor is practically constant, the injected e.m.f. is proportional to the rotor current of the induction motor. Further, the phase angle between the rotor current and injected e.m.f., i.e. between the resistance drop and rotational voltage is constant, and can only be changed by altering the construction of the machine. The diagram for this case is shown in Fig. 7.33.

Since $FG \propto AG$, and the angle FGA is constant, angle FAG is constant.

Also AF is at right angles to AD.

$$\therefore \qquad \angle GAD = 90° - \angle FAG = \text{constant}$$

But AG is parallel to BD, i.e. I_2,

$$\therefore \qquad \angle GAD = \angle BDA = \text{constant}$$

and, since AB is of constant length, it follows that the locus of D is a circle. The centre of the circle is easily found, since the circle

passes through A, B, and D. Bisect AB and BD, and draw perpendiculars to each through the points of bisection. Then the perpendiculars intersect at the point O, the centre.

Points A, B, and C are determined, as for Fig. 7.31, but now for X_2 we must substitute $X_2 + X_c + X_{cs}$, where X_c = leakage reactance, per phase, of the regulating motor, at primary frequency, and

$$X_{cs} = \frac{\text{volt-amperes to excite regulating motor}}{I_2^2 \times \sqrt{3}} \quad . \quad (7.233)$$

It is usually permissible to neglect saturation, but if present, it reduces X_{cs} and gives us a new and larger circle, but the ratio of injected e.m.f. to rotor current or exciting current is reduced. The

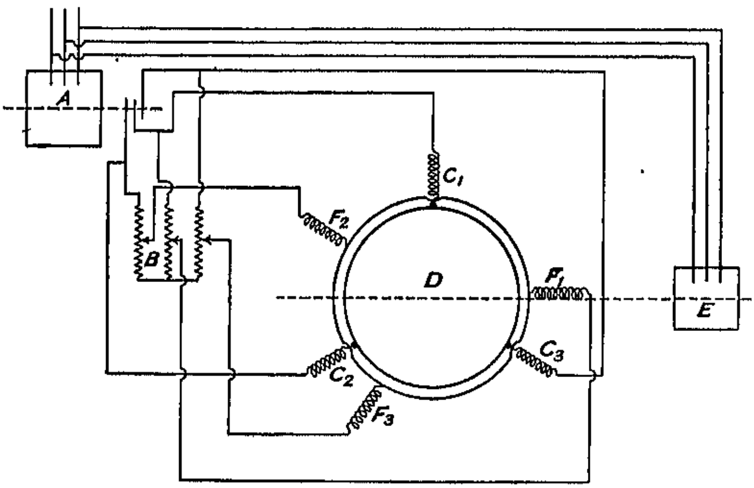


FIG. 7.34

two effects partially offset one another. In any case, it is simple to draw the new circle, provided we can determine the excitation, at full frequency, from the magnetizing current of the regulating motor.

The series regulating motor is most suitable for cases where large and rapidly fluctuating loads occur and where it is desirable to reduce the peak loads by means of a flywheel.

With a constant load, the speed may be regulated by hand, as desired, if the field windings of the commutator motor are supplied through a suitable transformer with taps.

Speed Regulation and Power-factor Control by Means of a Three-phase Shunt Commutator Motor

For some purposes speed variation is required but the speed should not vary greatly with the load. If the flux of the commutator motor can be adjusted to a constant value, the injected e.m.f. will be

practically constant, since the speed of the commutator motor is practically constant. By adjusting the magnitude of the field, the rotational e.m.f. can be given any suitable value, and this e.m.f. will be independent of the load. The field coils are fed from taps on the auto-transformer B (*see* Fig. 7.34). When the voltage across the rings increases, the slip frequency increases at the same rate and, therefore, the flux in the auto-transformer is independent of the slip. For the same reason the flux in F_1, F_2, and F_3 is constant for a given tap position and independent of the slip. It can be changed, however, by altering the tapping point on B. The circle diagram for this case is shown in Fig. 7.35.

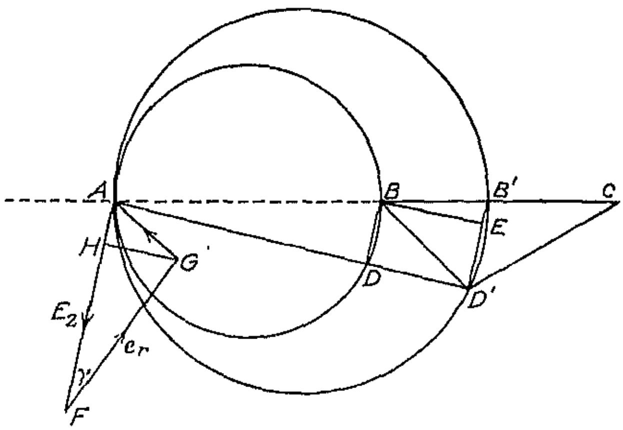


FIG. 7.35

With the commutator motor stationary, we get the circle BDA. In constructing this we must include the leakage reactance of the regulating motor, so that $X_{2+c} = X_2 + X_c$, and for C_2 we have C_{2+c}. With the commutator motor running, we get secondary current BD', and the total induced e.m.f. of the induction motor rotor AF is proportional to AD' and the slip. FG, the injected e.m.f. of the commutator motor, is proportional to AD' and is at constant angle to AF. This angle is determined by the connections of the transformer and exciting windings F_1, F_2, and F_3.

Resolve the resistance drop GA into two components: HA in phase opposition to AF and GH in quadrature to AF; the corresponding components of secondary current being BD and DD'. DD' is proportional to HG.

$HG = FG \sin \gamma$, so that $HG \propto FG$ for constant γ; but $FG \propto AD'$, therefore, $DD' \propto AD'$ and $AD = AD' - DD'$; therefore, AD is proportional to AD', therefore, $\dfrac{AB'}{AB} = \dfrac{AD'}{AD} =$ constant and B is a fixed point.

Therefore, the locus of D' is a circle.

When running light, i.e. zero torque, BD and AH are zero. The torque of the motor is proportional to the product of mutual flux and the component of secondary current in quadrature with it, i.e. to $BD \times AD$. It is zero, when BD is zero. BD is the torque producing component of BD'. The slip, when running light, $s_0 = \frac{FH}{AD'}$, and the additional slip, due to the load $= \frac{AH}{AD'}$

The running-light slip is determined by the angle γ and the ratio of $\frac{FG}{AD'}$, which conditions are adjusted by the connections at the auto-transformer. The load slip s_1 is the same for all values of s_0, provided the angle γ is chosen so that $\frac{HG}{AD'}$ remains constant.

The power factor can obviously be improved and the maximum torque of the motor increased.

Characteristics, intermediate between those obtained by means of a series and shunt a.c. commutator motor, can be obtained by means of a compound-excited neutralized commutator motor. It is necessary, in this case, to change the field voltage of the commutator motor, as the load changes, in order to change the flux and hence the injected e.m.f. This is effected by a series transformer, which has an air-gap in its magnetic circuit, so that its flux is proportional to the resultant of the primary and secondary ampere-turns.

In those cases in which speed regulation is required, below and above synchronism, a special exciter or frequency changer is required. This case has the advantage that, for a given range of speed regulation, a regulating set of only half the output of that required for a full-speed range below synchronism is needed.

A second important advantage is that the synchronous speed of the main motor is in the middle of the speed range, so that many processes may be carried out running as a plain induction motor, with the set shut down.

The special exciter is simply a rotor mounted on the shaft of the main motor. It is provided with a closed-circuit winding, connected to slip-rings on one side and to a commutator on the other, just like the armature of a rotary converter. It is wound for the same number of poles as the induction motor and the commutator has three brush sets per pole pair. The magnetic circuit of the armature is completed by a ring of laminated steel, placed over the slots and rotating with the armature. The slip-rings are connected with the mains, in such a manner that the field rotates in a direction opposite to that of rotation of the rotor. Since the speed of the field relative to the rotor is that of synchronism, the brush p.d. on the commutator (neglecting the drop in the rotor windings) will be practically constant. The frequency of the brush p.d. will be equal to the slip frequency of the main motor. This is a frequency changer, changing from supply

frequency to slip frequency. In passing through synchronism the phase sequence of the brush p.d.s is automatically reversed. Now this exciter is used to excite the fields of the commutator motor. Imagine that, from a suitable external source, whose frequency is always that of slip frequency, we excite the commutator motor so as to reverse the phases of the e.m.f.s, which it generates before they are reduced to zero. These e.m.f.s will be in phase with the rotor e.m.f.s, and the immediate effect is to increase the rotor currents and accelerate the rotor. As the speed increases, the rotor e.m.f.s of the induction motor decrease, and the current decreases till it reaches the value required to produce the load torque. The rotor current is now only partially maintained by the rotor e.m.f.s and is partially maintained by the commutator machine. The commutator machine is now acting as a generator and is driven by the induction machine coupled to it.

If the excitation of the commutator machine is now increased, the speed of the induction motor will rise and reach synchronism. At synchronism the induction-motor rotor e.m.f. is zero, and the rotor current is produced entirely by the e.m.f. of the commutator machine. This current, of zero frequency, is a continuous current. If this current is still further increased, the speed will rise above synchronism and the induction motor rotor will now generate e.m.f.s of opposite phase sequence and of reversed phase. Let us suppose that, as the main motor passes through synchronous speed, the phase sequence of the source, supplying the excitation of the commutator motor is reversed automatically, but without phase reversal of the e.m.f.s. The commutator motor will have the same sequence of e.m.f.s as the e.m.f.s in the rotor of the induction motor, but the latter will oppose the former. The rotor currents flow in the direction of the commutator e.m.f.s and, therefore, this machine acts as a generator, supplying the rotor copper losses of the induction motor, and also additional power, which is converted into mechanical power. The induction motor is now doubly fed, both stator and rotor receiving power from external sources. With increasing excitation of the commutator motor the speed of the induction motor will rise.

It is clear, therefore, that some source of excitation for the commutator machine is required, whose frequency is slip frequency and whose phase sequence is reversed as the motor passes through synchronous speed.

Fig. 7.36 shows the diagram of connections for double-range speed regulation. The field coils are shown as F_1, F_2, and F_3. They are fed from taps on the auto-transformer B, whose terminals are connected across the slip-rings of the induction motor. The inner ends of the field coils F are connected through the rheostats M to the commutator brushes of the frequency changes.

When it is required to change the speed, the flux is increased by

altering the tappings. The reactance drop component of the impedance drop of the field circuit, being proportional to the frequency as well as to the flux, is proportional to the square of the slip, while the resistance drop is proportional to the field current and slip.

By connecting to taps of B, whose distance from the star point is proportional to the slip, we get a voltage proportional to the

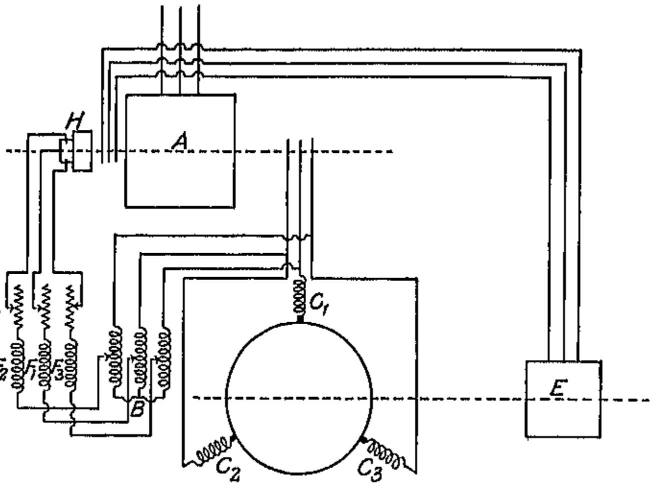


Fig. 7.36

square of the slip, since the total e.m.f. of B is proportional to the slip. By changing the taps on resistance M, so that the entire resistance of the circuit is proportional to $\frac{1}{\text{slip}}$, we first permit the constant-voltage frequency changer to supply the resistance-drop balancing e.m.f., while the auto-transformer B furnishes the reactance drop balancing e.m.f. One set of switches can be arranged to vary M and B at once.

The reader is referred to an article in the *Journal of the American Institute of Electrical Engineers* for June, 1920, and to the *E.T.Z.* for 1913, for further information.

Plate VIII (facing page 97) and Plate IX (facing page 212) show Ilgner sets for rolling mills.

CHAPTER VIII

The Three-phase Series and Shunt Commutator Motors

In Chapter VII concatenation of induction motors with the three-phase series, and three-phase shunt commutator motors was investigated. It will be of interest to give some more detailed account of these motors.

THE THREE-PHASE SERIES COMMUTATOR MOTOR

The series motor is due to the late Prof. Wilson, who took out a patent in 1888, and it also is attributed to Goerges, who took out a German patent in 1891. This machine has a stator similar, in all respects, to that of the three-phase induction motor. The rotor is similar to the armature of a d.c. machine and is provided with a commutator. The armature may be lap or wave wound. It has three sets of brushes per pair of poles; brushes of equi-potential are joined together to form a single rotor terminal. The rotor is connected in series with the stator, as shown in Fig. 8.1. The rotor is supplied through a series-type induction regulator transformer, by which means it is possible to obtain a whole range of speed-torque characteristics. Where a series transformer, with a fixed secondary is used, control of speed is effected by movement of the brushes.

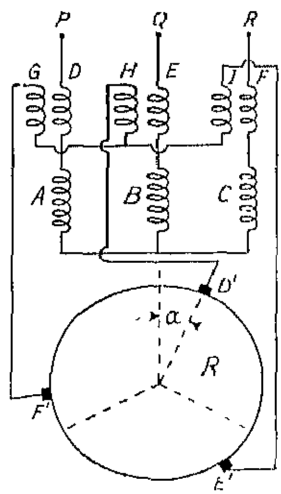


FIG. 8.1

Assume that this motor is supplied with a.c. current of frequency f. The stator current will, as in the induction motor, produce a revolving field. So also will the current in the rotor produce a revolving field. The actual revolving field in the machine is produced by the resultant of the stator and rotor ampere-turns. The magnitude of the resultant is determined, not only by the magnitude of the ampere-turns of stator and rotor, but by the position of the brushes on the commutator. That brush position, for which the ampere-turns

of stator and rotor directly oppose each other, is called the "neutral position" of the brushes. It is the datum position, from which the angle of brush shift is measured.

The direction of the torque depends on the direction of movement of the brushes from the datum position. Let us call the angle of brush shift from the datum position α. When $\alpha = 0$ there is no torque, for there is no displacement of stator and rotor fields. For a clockwise movement of the brushes, from the datum or neutral position, we have clockwise rotation, and vice versa.

In Fig. 8.1, A, B, and C represent the three stator phases. D, E, and F are the primaries of the three-phase series transformers. G, H, and I are the secondaries of the three-phase series transformer. P, Q, and R are three-phase supply terminals, while D', E', and F' represent the three brush sets on the commutator of the rotor.

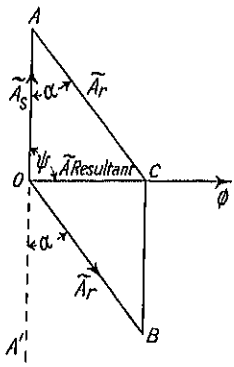


Fig. 8.2

In Fig. 8.2, OA represents the stator ampere-turns per pole, and OB the rotor ampere-turns per pole.

When $\alpha = 0$, OB is opposite to OA, and the brushes are in the neutral position. OA and OB oppose each other in this position.

Let the brushes be moved through an angle α counter-clockwise, then OB, the rotor ampere-turns per pole, makes an angle α with OA', the neutral position.

OC = resultant ampere-turns per pole.

Neglecting hysteresis, the flux per pole is in phase with OC.

$$OC = \sqrt{A_s^2 + A_r^2 - 2A_sA_r \cos \alpha} = A_R \quad . \qquad (8.1)$$

Now
$$\frac{A_R}{\sin \alpha} = \frac{A_r}{\sin \psi} \quad . \quad . \quad . \qquad (8.2)$$

$$\therefore \qquad \sin \psi = \frac{A_r \sin \alpha}{A_R} \quad . \quad . \quad . \quad . \qquad (8.3)$$

Vector Diagram for Three-phase Series Motor, Omitting Regulator

The resultant m.m.f. OC produces the flux ϕ, in phase with OC (neglecting hysteresis), and the e.m.f. which it induces in the stator winding lags the flux ϕ by 90°. Now, if the brushes are moved through an angle α, it is clear that the e.m.f.s, induced in the rotor, will be altered in time phase by the amount of the angle of brush shift, for the revolving field will cut the phase winding later with a shift of the brushes in the direction of rotation of the field.

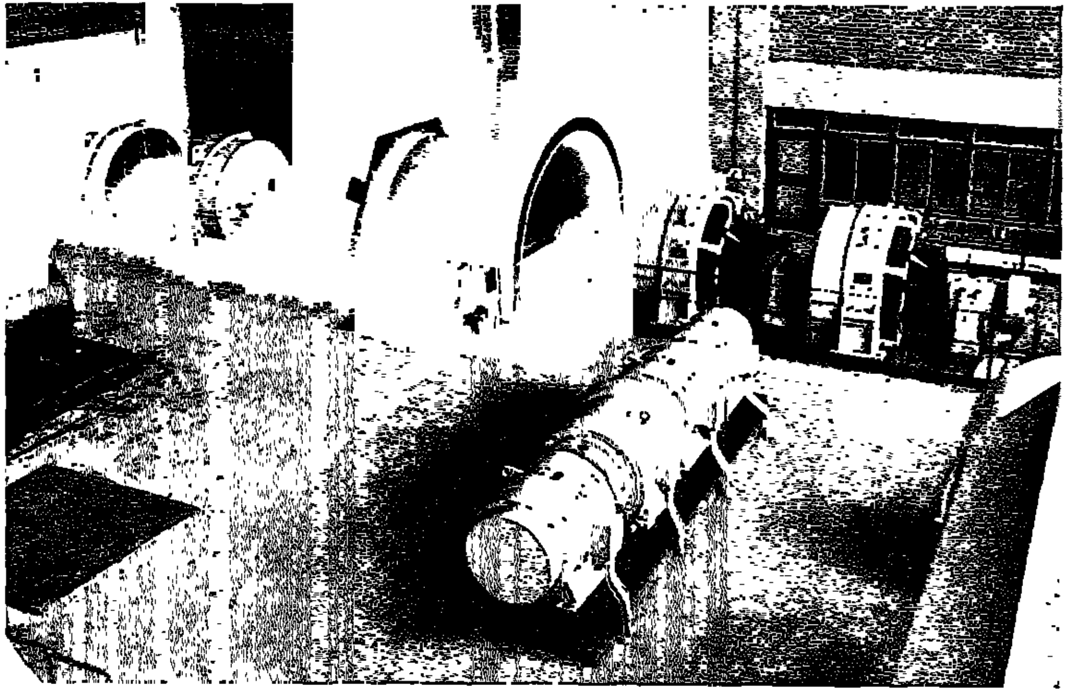
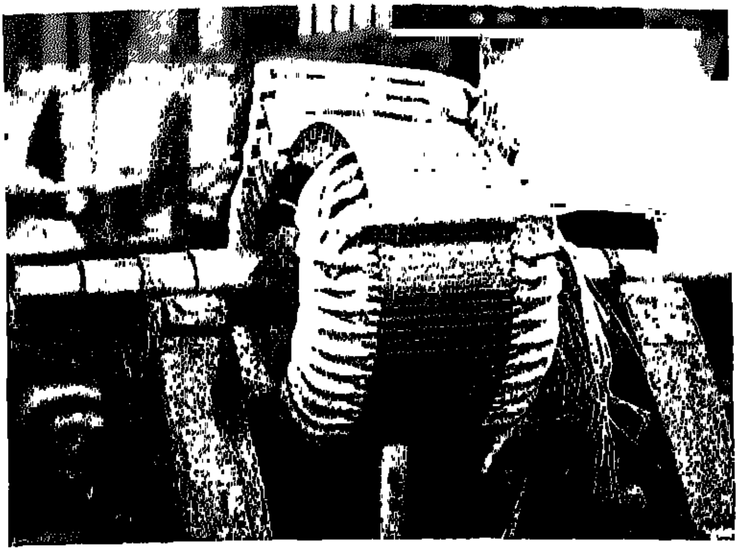


Plate IX. Ilgner Set with Exciter Set for Reversing Rail-mill Motor

(Courtesy English Electric Co. Ltd.)



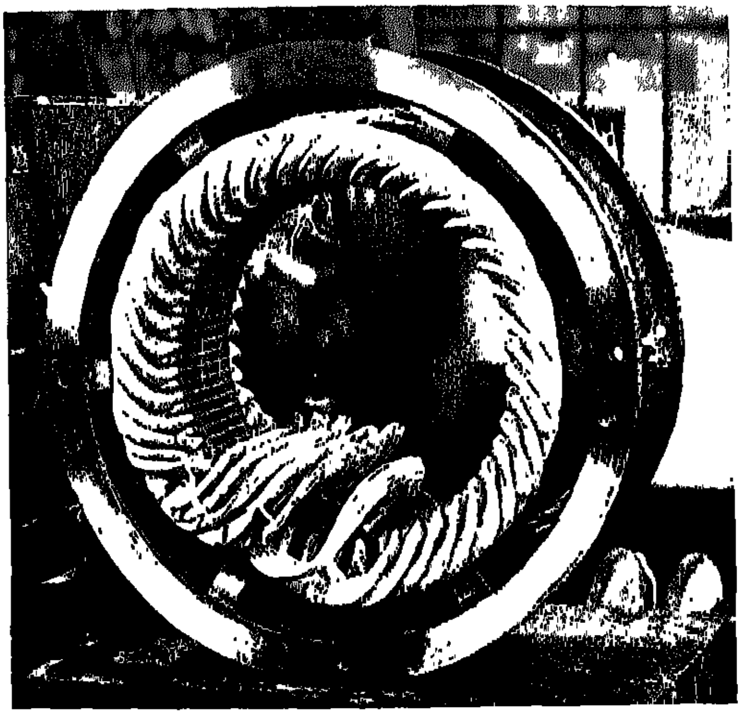


Plate X

(*Upper*) Mush-wound Rotor in Process of Winding

(*Courtesy Clarke, Chapman & Co., Ltd.*)

(*Lower*) Mush Winding for the Stator of a Three-phase Motor, in Process of Winding

(*Courtesy Clarke, Chapman & Co., Ltd.*)

In Fig. 8.3, OA = stator ampere-turns per pole and also the phase of the current

OB = rotor ampere-turns per pole; the angle $AOB = \pi - \alpha$

α = angle of brush shift from the neutral position

OC = resultant ampere-turns per pole, and phase of the flux ϕ. This is a combined space and time diagram.

$E_s = OF$ = e.m.f. induced in the stator per phase, and lags $\frac{\pi}{2}$ behind ϕ

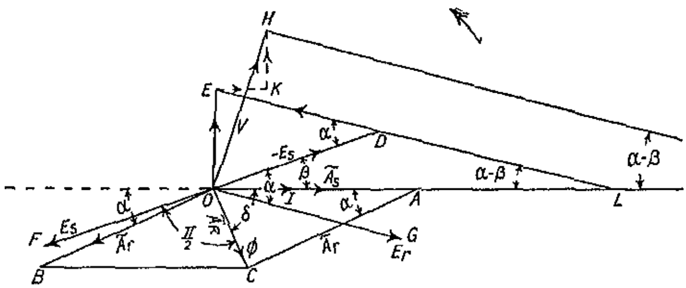


FIG. 8.3

$OD = OF = -E_s$ = component of supply volts to overcome E_s

OG = e.m.f. induced in the rotor per phase by the flux ϕ; it makes an angle $180 - \alpha$ with E_s

DE = equal and opposite to OG is the component of supply volts to overcome E_r

OE = resultant of OD and DE

EK = total resistance drop in stator and rotor, in phase with I, i.e. with OA

KH = component of supply volts to overcome the sum of the *leakage* reactance voltages in the machine

and OH = supply volts per phase

There should be a vector from the point E drawn parallel to the current vector $I(R_s + R_r')$, and a vector $I(X_s + X_r')$ leading the current by 90°. Here R_s = resistance of the stator per phase; R_r' = resistance of the rotor per phase, referred to the stator; X_s is the leakage reactance of the stator; and X_r' is the leakage reactance of the rotor per phase, referred to the stator. Since the same value of I is used for the drops, it is assumed that the

delta-connected armature is replaced by an equivalent star-connected winding. Then the applied volts per phase $= OH = V$.

The e.m.f. E_s, generated in the stator by the revolving field ϕ

$$= 2{\cdot}22 \times \hat{\phi} \times Z_s \times f \times 10^{-8} \times K_1 \times K_3 = E_s \quad . \quad (8.4)$$

where $\hat{\phi}$ = flux per pole (maximum value)

Z_s = number of conductors in series, per phase, on the stator

f = supply frequency

$$K_1 = \text{breadth factor for the fundamental} = \frac{\sin q_1 \frac{\lambda_1}{2}}{q_1 \sin \frac{\lambda_1}{2}}$$

where q_1 = slots per pole, per phase, in the stator

λ_1 = electrical slot pitch angle in the stator

$= \frac{180}{3q}$ for a three-phase machine

$K_3 = \cos \frac{\varepsilon}{2}$ = stator coil span factor for the fundamental

where ε = deficiency of pitch of coil from full pitch in electrical degrees

$$\text{Also } E_r = 2{\cdot}22 \times K_2 \times K_4 \times s \times f \times Z_2 \times \hat{\phi} \times 10^{-8} \quad . \quad (8.5)$$

where E_r = generated volts per phase in the rotor

s = slip

K_2 = rotor breadth factor

$$= \frac{\sin q_2 \frac{\lambda_2}{2}}{q_2 \sin \frac{\lambda_2}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.6)$$

K_4 = coil span factor in the rotor

q_2 = slots per pole per phase in rotor

λ_2 = slot pitch angle in electrical degrees in rotor

It is clear that as the current varies OA and OB will vary proportionately, and since for a given brush shift α, OB is fixed in direction, the resultant vector OC, which produces the flux is fixed with respect to OA, i.e. with respect to the current. The angle DOA, in Fig. 8.3, is fixed, and so also is the angle ODE, which is equal to α, the brush shift angle. Since OE is fixed and equal to the applied voltage per phase, neglecting resistance and leakage reactance voltages, and the angle ODE is constant, it follows that the locus of the point D is a circle, i.e. as the load and speed vary the locus of D is a circle, of

which OE is a chord. Produce ED to L, and the current vector to L, i.e. ED and OA produced intersect at L. Then the angle ELO is equal to $\alpha - \angle DOA = \alpha - \beta$. But the angle DOA is a fixed angle, as we have shown, therefore, the locus of L, when the load and speed vary for a given brush shift α, is another circle through the points O, E, and L, neglecting resistance and leakage reactance drops.

In Fig. 8.4 is shown the circular current locus, viz. OLE and the circle of voltages, both neglecting resistance drops and leakage

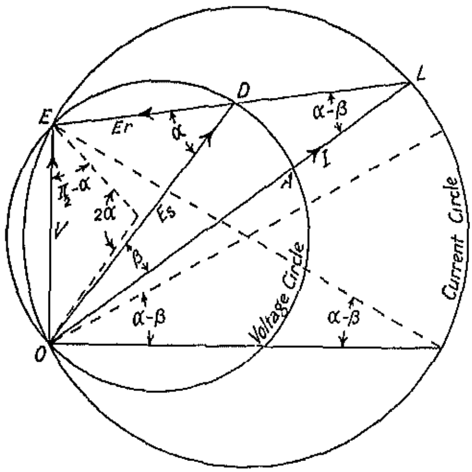


Fig. 8.4

reactance drops. When D coincides with E synchronous speed is reached; $E_r = 0$ and $E_s = V$. The radius of the voltage circle is given by the equation

$$2R_1 \sin \alpha = V \quad . \quad . \quad . \quad . \quad (8.7)$$

$$\therefore \qquad R_1 = \frac{V}{2 \sin \alpha}$$

At standstill $s = 1$,

$$\frac{E_r}{E_s} = \frac{K'_2 \times K'_4 \times Z_2}{K'_1 \times K_3 \times Z_s} \quad . \quad (8.8)$$

where Z_2 = conductors in series per phase in the rotor in the equivalent star winding.

From standstill to synchronous speed the point D, in Fig. 8.4, moves around the voltage circle, from the standstill point, in a counter-clockwise direction till it reaches the synchronous speed point E. Further movement, in a counter-clockwise sense, of the point D around the voltage circle brings the speed range above synchronism, until we reach twice synchronous speed; at which point E_r has the same value as at standstill.

The radius of the larger current circle $= R_2$,

where $$R_2 = \frac{V}{2 \sin (\alpha - \beta)} \qquad (8.9)$$

The current scale can be determined from the value of E_r at standstill ($s = 1$), or from the normal value of ϕ the flux per pole at full load and the open-circuit characteristic. It is clear that, below synchronous speed, the stator absorbs energy from the line, while the rotor gives energy to it. At synchronous speed the stator alone absorbs energy. Above synchronous speed the stator and rotor absorb energy, which is converted into mechanical energy. The torque is proportional to the product of flux per pole and the armature ampere-turns $\times \sin \alpha$,

$$= KI^2 \sin \alpha$$

The torque is a maximum when the rotor ampere-turns make an angle of 90° with the resultant ampere-turns OC in Fig. 8.3, i.e. with ϕ.

When $\angle OCA$ in Fig. 8.3 is a right angle, we have

$$\bar{A}_s \cos \alpha = \bar{A}_r$$

also $$\frac{\bar{A}_{\text{Resultant}}}{\bar{A}_r} = \tan \alpha \qquad (8.10)$$

Let $$m = \frac{\bar{A}_r}{\bar{A}_s} = \frac{\text{ampere-turns rotor}}{\text{ampere-turns stator}} = \cos \alpha \qquad (8.11)$$

Maximum torque occurs when $\bar{A}_R$ and $\bar{A}_r$ are at right angles

$$= K\bar{A}_R \times \bar{A}_r$$
$$= K\bar{A}_s \sin \alpha \times \bar{A}_s \cos \alpha$$

Therefore, maximum torque

$$= K\bar{A}_s^2 \sin \alpha \cos \alpha$$

$$= K\bar{A}_s^2 \frac{\sin 2\alpha}{2} \qquad (8.12)$$

i.e. maximum torque occurs when $\alpha = 45°$ or the angle AOB 135° (*see* Fig. 8.2).

Then, with this value of the brush shift

$$\frac{\bar{A}_r}{\bar{A}_s} = \cos 45 = \frac{1}{\sqrt{2}} \qquad (8.13)$$

i.e. for maximum torque the ratio of

$$\frac{\text{rotor ampere-turns}}{\text{stator ampere-turns}} = \frac{1}{\sqrt{2}} \qquad (8.14)$$

For this ratio of $\dfrac{\text{rotor ampere-turns}}{\text{stator ampere-turns}}$ the power factor, which is the cosine of the angle HOA, is low.

For this maximum torque condition, $\bar{A}_R = \bar{A}_r \tan \alpha = \bar{A}_r$ for $\alpha = 45°$.

For good power-factor conditions, usual values for

$$\frac{\bar{A}_R}{\bar{A}_r} \simeq 0{\cdot}6 \text{ to } 0{\cdot}4 \text{ and } \alpha = 22° \text{ to } 30° \quad . \quad . \quad (8.15)$$

and

$$\frac{\bar{A}_r}{\bar{A}_s} \simeq 1{\cdot}17 \text{ to } 1{\cdot}08 \text{ with mean of } 1{\cdot}12$$

The angle β in Fig. 8.3 is the angle DOA

$$= \frac{\pi}{2} - \angle AOC = \frac{\pi}{2} - \delta$$

Now

$$\frac{\bar{A}_r}{\sin \delta} = \frac{\bar{A}_R}{\sin \alpha} \quad . \quad . \quad . \quad . \quad (8.16)$$

$\therefore$

$$\sin \delta = \frac{\bar{A}_r \sin \alpha}{\bar{A}_R} \quad . \quad . \quad . \quad . \quad (8.17)$$

Clearly by making δ large and $> \frac{\pi}{2}$, then OD falls below OA, and it is clear that the power factor is greatly improved.

The whole series of characteristics can be plotted from Fig. 8.4 by taking various positions for D on the circle, for speeds below and above synchronous speed.

Thus, for a given value of ED, i.e. E_r, from the e.m.f. equation we can determine $s \times \phi$, and from the corresponding value for E_s, viz. OD, we can determine ϕ from the e.m.f. equation. Thus, the slip can be determined and the speed $= \omega_0(1 - s)$. Also, the torque, current, slip, input, and efficiency can be readily determined.

Characteristic Curves for the Three-phase Series Commutator Motor

Characteristic curves are shown in Fig. 8.5 for the series type of motor, from which the behaviour of the motor can be seen for speeds up to 1·5 times the synchronous speed.

One must draw a similar diagram for every brush position to that of Fig. 8.4. The angle α of brush shift is a variable parameter, which is held fixed for one set of curves; by varying α various sets of such curves are obtained. The part BC of the torque curve (Fig. 8.5) represents unstable conditions. Thus, the torque falls with the speed near standstill. This part of the curve can be modified by suitable design.

This motor has admirable characteristics for applications such as hoists, winches, etc. It resembles, in many ways, the d.c. series motor, but instead of one torque-speed curve being obtained, as in the series d.c. motor, with the three-phase series type, a whole series of such curves are obtainable by brush shift. Instead of shifting the brushes, it is usual to use a series induction regulator, which is a rotary current transformer with movable secondary. Thus, the phase of the applied voltages to the rotor can be varied by moving the secondary of the regulator. This effects the same results as shifting the brushes on the commutator, but it also reduces the voltage on the armature, which is necessary for the reason which we now develop.

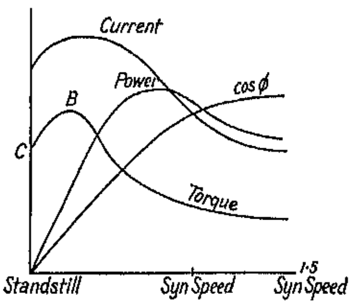


FIG. 8.5

The great drawback with these motors is the difficulty involved in commutation. In addition to the reactance voltage in the coil undergoing commutation, which is present in the d.c. machine, and which is due to the change of leakage flux as the current in the coil changes, when it passes the brush, i.e. when the coil passes the brush, the current in the coil changes from the value in one phase to that of the consecutive phase, there is an additional voltage, generated in the coil, due to the rotating field. This voltage, usually known as the transformer voltage, has its maximum value at standstill, viz.

$$e_t = 4{\cdot}44 \times \hat{\phi} \times T_c \times sf \times 10^{-8} \quad . \quad . \quad (8.18)$$

where T_c = turns per coil

s = slip

f = supply frequency

Its value is given in equation (8.18).

Now it is clear that unless special precautions are taken, such as the use of the Schwarz patent winding, or the use of another scheme by the author (now being patented), this transformer voltage must be kept low enough to allow good commutation to be obtained. In this case high contact-resistance brushes must be used and high-resistance connections between the coils and the commutator. This transformer voltage in the coil must, therefore, not exceed 3 V, approximately. This can only be obtained by using the following—

(*a*) a small flux per pole;
(*b*) single-turn coils in the armature;
(*c*) a low supply frequency;
(*d*) a large number of poles.

It will be seen that the voltage which can be supplied to the armature is limited, for on a given diameter of commutator, which is limited by the peripheral speed permissible, there are three sets of brushes per pole pair. On an eight-pole machine there will be twelve brush sets, and if we allow, say, 6 in. for the circumferential distance between brush sets, this will allow us about thirty segments between brushes, or about 100 V supply voltage. The diameter of the commutator, in this case would be 23 in. and the peripheral speed of the commutator about 4500 ft/min at 750 r.p.m., assuming the machine is running at synchronous speed on 50 c/s.

It is a characteristic of these machines to have a large diameter of commutator, with a large number of segments. A limit is set to the diameter by the peripheral speed of both armature and commutator, and the higher this is, the more difficult do the commutating conditions become. It must be remembered, too, that speeds of two or three times synchronous speed, or more, may be required. Thus, not only is the voltage supply to the commutator limited in magnitude, but the electric loading of the armature is also limited by the diameter limitation. It follows, therefore, that there is a definite limit to the output *per pole pair*, of the order of about 20 kW. This does not apply with the auxiliary winding used by Dr. Schwarz. With the Schwarz winding the limit to the voltage per segment is removed, and there is not the need to use a small flux per pole. Indeed, Dr. Schwarz has removed the one great drawback to these machines and has placed them on a more or less equal footing with the d.c. machines, as far as commutation is concerned. With the usual design, these machines are characterized by several features, common to all a.c. commutator motors, viz. large diameter, short axial length, large commutator diameter, large number of brush sets, and small radial depth of iron above the stator teeth and below the rotor teeth.

We have already mentioned the various voltages which cause commutation trouble. Consider again the reactance voltage, due to current change in the coil as it passes under the brush.

This is the voltage which is present in d.c. machines, but in the a.c. commutator motor, the current change on passing under the brush is different from that in the d.c. machine. Before the coil passes under the brush, it carries current of one phase of the armature, say, that of phase 3 for example; when it passes the brush it passes into the consecutive section of the armature, which is carrying current of phase 1, for example. Thus, the currents in the sections of the armature between brushes are assumed to be sinusoidal and differ by 120° in time phase.

The current will follow the values of the sine curve of one phase till it reaches the brush. After leaving the brush, the current, in the coil considered, must have the value appropriate to the values carried by the branch of the armature, into which it has entered.

Thus, the amount by which the current changes, during commutation, is equal to the difference of the instantaneous values of the current in two adjacent sections of the armature and is, therefore, equal to the instantaneous value of the brush current. The maximum change of current in the coil is equal to the maximum brush current. The e.m.f. of self-inductance in the short-circuited coil, due to the current change may be regarded as in phase with the brush current. The time of commutation, which is the time the coil is short-circuited by the brush, depends on the width of brush and the peripheral speed of the commutator and is very short, probably between $\frac{1}{1000}$ and $\frac{2}{1000}$ sec.

This e.m.f., due to current change in the coil undergoing commutation, is similar to that in the d.c. machine. It retards commutation. In the d.c. machine it is neutralized by a rotational e.m.f., generated by rotation in an interpole flux, and the same is also true in the a.c. single-phase commutator motor. This e.m.f. can be kept down to a low enough value by (*a*) using single-turn coils in the armature; (*b*) keeping the electric loading to a sufficiently low value; (*c*) using narrow brushes, so as not to short-circuit more than one coil at any time; and (*d*) keeping down the peripheral speed of the commutator.

The voltage, however, which is the most serious in this machine and which has been responsible for its limitations, both in design and application, is the voltage which is generated in the short-circuited coil by the *main* flux. At standstill this voltage is a maximum and becomes zero at synchronous speed.

Its value is given by equation (8.18).

The Schwarz Winding

To Dr. Schwarz we are indebted for one solution of the problem of the voltage generated in the short-circuited coil by the main flux. He uses another winding, called the "auxiliary" winding, in the same slots as the main winding. The main winding is usually an ordinary d.c. lap winding connected to commutator segments. The armature slots are deep, and between the main winding and the auxiliary winding, in the same slots, are placed sheet-iron interleaves, which form an additional magnetic path for the auxiliary field of the auxiliary winding. The auxiliary winding is a lap winding, connected to the same commutator segments as the main winding, but the auxiliary winding coils have twice as many turns as the coils of the main winding, but the coil span of the auxiliary coils is but one-third of the pole-pitch. The coil span factor for the auxiliary winding is $\cos 60 = \frac{1}{2}$, and so, for the same voltage per segment, the number of turns per coil must be twice that of the main coils. The two windings—main and auxiliary—are connected in parallel to the same commutator segments. The arrangement is shown in

Fig. 8.6. In Fig. 8.6 the auxiliary coils x_1 and x_2 are in the same slots, and form a transformer link with almost perfect magnetic coupling; with common fluxes ϕ_h' and ϕ_h''. x_2 is connected with segments 2 and 3, and is, therefore, in parallel with the main winding b_1, which is on the left of slots Nb and Nb'; x_1 is connected in parallel with a_2 and lies in slots Nx and Nx'. The energy of the leakage

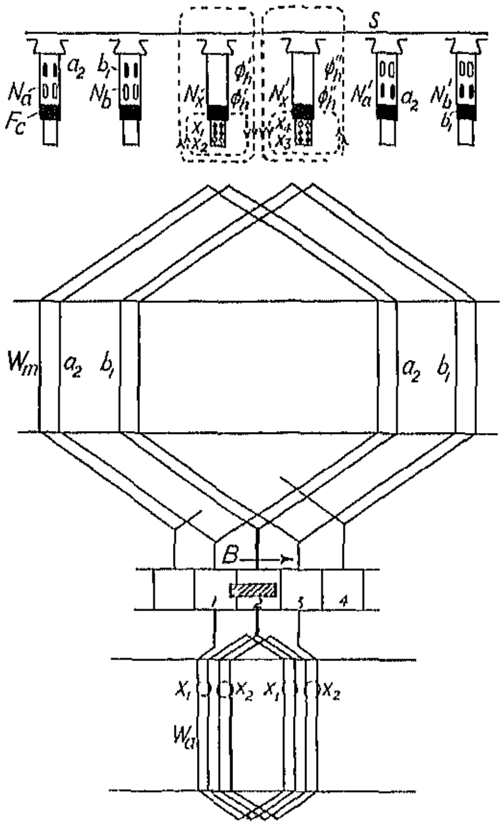


FIG. 8.6

field, which produces the sparking, is transferred from the coil a_2 to x_1 and x_2 and to b_1. The reactance of the winding a_2 and the reactance of b_1 are almost halved by the assistance of the transformer x_1 and x_2. Since b_1 lies in the same slot as b_2, there exists a transformer linkage, which continues through the next auxiliary groups.

In the words of the inventor: "In this way a parallel connection of windings is obtained in both directions round the armature, so that the operative reactance of the winding, while commutating is reduced to a fraction of its normal value."

This arrangement operates satisfactorily at the interruption of

the transformer currents, during the commutation period. The net result is that the voltage per segment can be increased until it is no longer the voltage, but the losses produced by the transformer currents that determine the maximum limit for the segment voltage. The inventor has found it possible to increase the current density in normal graphite brushes from 7 or 8 A/cm² to 12 or 14 A/cm², without harm. Thus, he is able to use large flux per pole, reasonably large voltage per segment, and to reduce the size of the commutator. There is little doubt that this invention has largely removed the limits to design and application.

In some cases intermediate segments between the main segments are provided for the auxiliary winding. In Fig. 8.7 one turn of the main winding is shown, and the winding group of the auxiliary winding, which is connected in parallel with it. The main winding is connected to segments 1 and 2, and so are the beginning and end of the auxiliary winding, which consists of two windings in series. The connection between the first and second auxiliary windings is brought out and connected to segment 1′. The voltage between segments 1 and 2 is divided into two equal parts at segment 1′. The main current now flows through 1′, and the two auxiliary windings are connected in parallel, and have no reactance, since the magnetic effects of the component currents are cancelled.

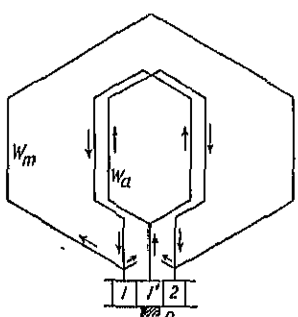


FIG. 8.7

By the use of these devices it is possible to build motors with a larger horse-power per pole.

The reader is referred to the paper by Dr. Ing. Benno Schwarz, Rheydt, in *Electrotechnik und Maschinenbau*, on "Recent Developments of the Stator-fed Polyphase Shunt-wound Commutator Motor," 24th February, 1935.

There is little doubt that this invention of Dr. Schwarz has profoundly affected these motors, removing limits which have, hitherto, prevented the more extensive use of these machines.

Another Method of Approach (due to Shuttleworth)

Shuttleworth has given another method of approach* which will, perhaps, give a clearer picture of this motor and its analysis. In this method one single-phase is used, which is permissible, if the other phases are not neglected in their effect. Briefly, he adopts the device,

* Shuttleworth, N.: "Polyphase Commutator Machines and their Applications," *Journal I.E.E.*, March (1915).

commonly used in the theory of the single-phase repulsion motor, of resolving the ampere-turns of the stator into two components—

(*a*) along the brush axis;

(*b*) at right angles to the brush axis (*see* Fig. 8.8).

The stator phase is displaced by angle α from the rotor phase and S = number of stator turns *per pole per phase* (effective)

R = number of rotor turns per pole per phase (effective)

p = number of poles

$\hat{I}$ = maximum current per phase

FIG. 8.8

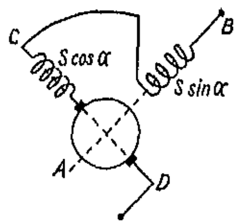


FIG. 8.9

Exciting ampere-turns along the brush axis $= K(S\cos\alpha - R)\hat{I}$, the factor K takes into account the effect of the other phases. $K = \frac{3}{2}$ for a three-phase machine.

It will be remembered that the amplitude of the resultant ampere-turns for m phases $= \frac{m}{2} \times$ amplitude for one phase. Now the turns $S\sin\alpha$ are called the exciting turns and the axis AB the "excitation" axis, and the corresponding flux along AB the "excitation" flux.

The total exciting ampere-turns $= KS\sin\alpha\hat{I}$.

The resultant of the two components of exciting ampere-turns, which are at right angles

$$= K\hat{I}\sqrt{S^2 + R^2 - 2SR\cos\alpha} \qquad . \qquad . \quad (8.19)$$

The flux per pole, due to this excitation, may be obtained from the open-circuit curve. If there is no saturation, then

$$\phi = K_0 K\hat{I}\sqrt{S^2 + R^2 - 2SR\cos\alpha} \qquad . \qquad . \quad (8.20)$$

where K_0 = constant

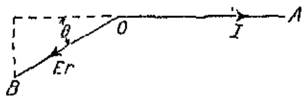


FIG. 8.10

If the currents in the exciting winding alone produced the field, the e.m.f. between brushes, generated by rotation, would be exactly opposite to the current, i.e. 180° out of phase. There is another excitation at right angle, viz. $S\cos\alpha - R$ per phase.

This causes the rotational voltage to differ from the phase of the current by $180 - \theta$, as shown in Fig. 8.10, here $\tan\theta = \dfrac{S\cos\alpha - R}{S\sin\alpha}$

Fig. 8.10 shows the relation of voltage and current.

$E_r = OB =$ rotational voltage.

The voltage induced in the exciting winding of the stator per phase

$$= \sqrt{2}\pi \times f \times \hat{\phi} \times pS\sin\alpha \times 10^{-8} \quad . \quad . \quad (8.21)$$

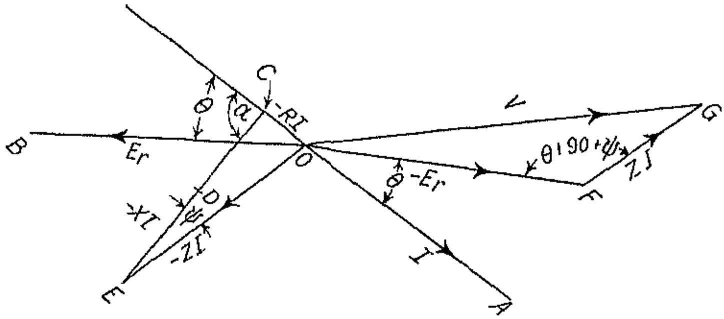


Fig. 8.11

The voltage induced in the rotor and the transformer axis of the stator winding

$$= \sqrt{2}\pi \times f \times \hat{\phi} \times p(S\cos\alpha - R) \times 10^{-8} \quad . \quad (8.22)$$

The resultant of these two voltage (which are at right angles)

$$= \sqrt{2}\pi \times \hat{\phi} \times f \times p\sqrt{S^2 + R^2 - 2SR\cos\alpha} \times 10^{-8} \quad (8.23)$$

where $\hat{\phi}$ is given by equation (8.20)

These voltages lag I by 90°.

To this resultant voltage we must add the leakage reactance voltages in stator and rotor.

In Fig. 8.11, $OB =$ counter e.m.f. in armature

$OA =$ current vector

$OC =$ resistance volts (total for stator and rotor)

$CD =$ sum of leakage reactance voltages in stator and rotor

$DE =$ vector sum of voltages, equations (8.21) and (8.22), and given by equation (8.23)

$OE =$ impedance voltage, due to self and mutual impedances

$OF = -E_r =$ component of supply volts to overcome E_r

$ZI = -OE = FG$

$OG =$ supply volts per phase $=$ constant

Now $OF = -E_r \propto$ speed of rotor $\times\ \phi \propto$ speed $\times$ current (if no saturation).

Now θ is constant for α constant and ψ is also constant.

Therefore, angle OFG is constant.

Therefore, the locus of F is a circle with $OG = V$ as chord.

The angle subtended by OG, at the centre $= 2[90 - (\theta + \psi)]$, and the radius of the circle R, for (θ positive),

$$= \frac{V}{2\cos(\theta + \psi)} \qquad (8.24)$$

Fig. 8.12 shows the circle surrounding the points O, G, and F, but θ is made negative to improve the power factor, but

$$\tan\theta = \frac{S\cos\alpha - R}{\sin\alpha} \qquad (8.25)$$

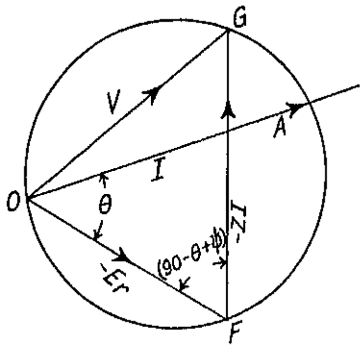


FIG. 8.12

Therefore, $R > S\cos\alpha$ when θ is negative.

Looking at Fig. 8.12, this will mean bringing OF below OA.

The radius of the circle (for θ negative)

$$= \frac{V}{2\cos(\theta - \psi)} \qquad (8.26)$$

At standstill OF is zero, and the current, which is proportional to FG, approaches a maximum value. As the speed increases, OF increases and GF decreases; I, therefore, decreases. At very high speed OF approaches OG and the current tends to zero. FG, taken to the proper scale, may represent the magnitude of the current, and the phase of the current is always θ with respect to $-E_r$ or OF. The value of the power factor is always known for all values of the current, for the current determines FG in Fig. 8.12, and once the point F is determined, $-E_r$ is known and I makes an angle of θ with $-E_r$ always.

It is clear that unity power factor, at normal speed, is obtainable by suitable design.

It is necessary to draw a separate diagram for each value of α, for θ depends on α, and each value of α gives a definite series of characteristics.

Now FG may be large, even with moderate value of the current. Since the reactance depends on $R^2 + S^2 - 2SR \cos \alpha$, when α is near 180°, this is large, when $\cos \alpha$ is negative. The current then is small, even at standstill. This is the usual starting position.

If we neglect the resistance drop ψ becomes 0 in Fig. 8.11 and the circle diameter

$$= \frac{V}{\cos \theta} \quad . \quad . \quad . \quad . \quad . \quad (8.27)$$

Let $\frac{R}{S} = m$

then
$$\tan \theta = \frac{S \cos \alpha - R}{S \sin \alpha} = \frac{\cos \alpha - m}{\sin \alpha} \quad . \quad . \quad (8.28)$$

$$\cos \theta = \frac{1}{\sec \theta} = \frac{1}{\sqrt{1 + \tan^2 \theta}}$$

$$= \frac{1}{\sqrt{1 + \frac{\cos^2 \alpha - 2m \cos \alpha + m^2}{\sin^2 \alpha}}} \quad . \quad . \quad (8.29)$$

$$= \frac{\sin \alpha}{\sqrt{1 + m^2 - 2m \cos \alpha}} \quad . \quad . \quad . \quad (8.30)$$

The torque is proportional to $\phi \times I \times \cos \theta$ and

$$\phi = KK_0 I \sqrt{R^2 + S^2 - 2RS \cos \alpha} \quad . \quad . \quad (8.31)$$

$$= K_2 IS \sqrt{m^2 + 1 - 2m \cos \alpha}$$

$$\therefore \qquad \underset{\text{torque}}{T} = \text{constant} \times I^2 \sin \alpha \quad . \quad . \quad (8.32)$$

This is a maximum when I is a maximum, i.e. when FG is a maximum in Fig. 8.12. FG is obviously a maximum when it becomes a diameter of the circle, i.e.

$$= \frac{V}{\cos \theta} \quad . \quad . \quad . \quad . \quad . \quad (8.33)$$

but FG (neglecting resistance drop) $= X_T I$

where $X_T = X_L + X_M$

$$= X_L + K_3 \times [R^2 + S^2 - 2RS \cos \alpha] \quad . \quad . \quad (8.34)$$

where X_L = sum of leakage reactances of stator and rotor

X_M = mutual reactance of both

$$X_M I = \sqrt{2}\pi \times \hat{\phi} \times f \times p\sqrt{S^2 + R^2 - 2SR\cos\alpha} \times 10^{-8} \quad . \quad (8.35)$$

$$= \sqrt{2} \times \pi \times K_0 K I \sqrt{S^2 + R^2 - 2SR\cos\alpha}$$

$$\times f \times p \times \sqrt{S^2 + R^2 - 2SR\cos\alpha} \times 10^{-8} \quad . \quad (8.36)$$

$$X_M = K_3 \times (R^2 + S^2 - 2RS\cos\alpha) \quad . \quad . \quad (8.37)$$

I_{max} = maximum r.m.s. current

$$= \frac{V}{\cos\theta[X_L + K_4(m^2 + 1 - 2m\cos\alpha)]} \quad . \quad . \quad (8.38)$$

where
$$K_4 = K_3 S^2 \quad . \quad . \quad . \quad . \quad . \quad (8.39)$$

maximum torque

$$= \text{constant} \times I_{max}{}^2 \sin\alpha$$

$$= \text{constant} \times \frac{V^2}{\cos^2\theta[X_L + K_4(m^2 + 1 - 2m\cos\alpha)]^2} \quad (8.40)$$

THE THREE-PHASE SHUNT COMMUTATOR MOTOR

We have already seen that the three-phase series motor varies its speed with the load, in a similar manner to the d.c. series motor. It has long been recognized, however, that a variable speed three-phase motor which has a shunt characteristic is desirable, i.e. when a given speed is obtained, it is desired that the speed shall not vary much with the load. In this respect, the three-phase shunt motor resembles the d.c. shunt motor. This motor has an outward appearance, similar to the three-phase series motor, i.e. a stator with a three-phase winding, like an induction motor, and a rotor with an ordinary d.c. winding connected to a commutator. The armature winding is usually of the lap type. The commutator has three brush sets per pole pair; similar brush sets are joined together to form a single terminal. There are three such terminals.

Both stator and rotor are supplied in parallel, with voltages of the same frequency, but of different magnitude. To vary the speed, we have seen that a voltage must be impressed on the rotor winding, having the same direction and frequency as the e.m.f. induced in the winding by the rotating field. If the impressed secondary voltage is in phase with the induced e.m.f., the speed will rise, for a reduced slip is required to produce the current and torque. If the impressed e.m.f. is 180° out of phase with the induced e.m.f., the slip will increase and the speed fall. By adjusting the magnitude and phase of the impressed voltage, one has control of speed and power factor. The impressed voltage must have the same frequency as the e.m.f. induced in the rotor.

But, it may be asked, how can one supply the rotor with a voltage having the same frequency as the stator supply voltage, and at the

same time have slip-frequency voltage impressed on the rotor conductors?

The speed of the revolving field, in revolutions per minute

$$= \frac{120f}{p}$$

where f = frequency
and p = number of poles.

The frequency of the voltage generated at the brushes—*which are fixed in space—is the line frequency*, while the frequency of the e.m.f.s in the conductors of the rotor is of slip frequency, depending on the *relative motion* of field and rotor.

Thus, the brushes of a three-phase commutator motor may be connected to the same source of supply as the stator. A variable secondary voltage has generally been supplied by brush shifting, as in the Schräge motor. The constant secondary voltage of a single-stage rotary controller is not suitable for the whole range of speed, so it is usual with single-stage rotary controllers to provide brush shift in conjunction.

Brush shifting can be avoided by the methods described below.

The Single-stage Rotary Controller

In this single-stage rotary controller, the stator windings are A, C, E; B, D, and F are auxiliary windings, usually situated on the stator (*see* Fig. 8.13).

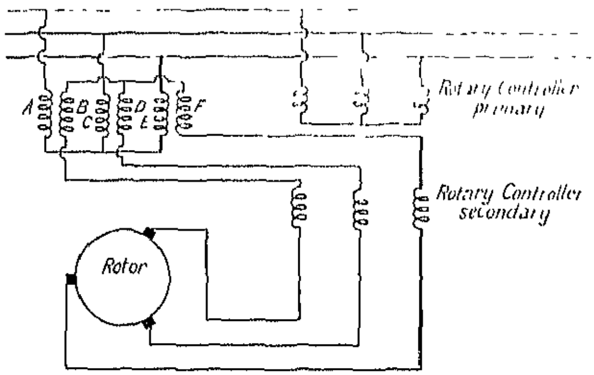


FIG. 8.13

The primaries of the stator, which are connected in star, and the primary of the rotary controller are connected to the supply lines. The secondaries of the rotary controller and the secondaries of the current transformer are connected in series to the rotor. The magnitude of the secondary voltage remains constant and only the

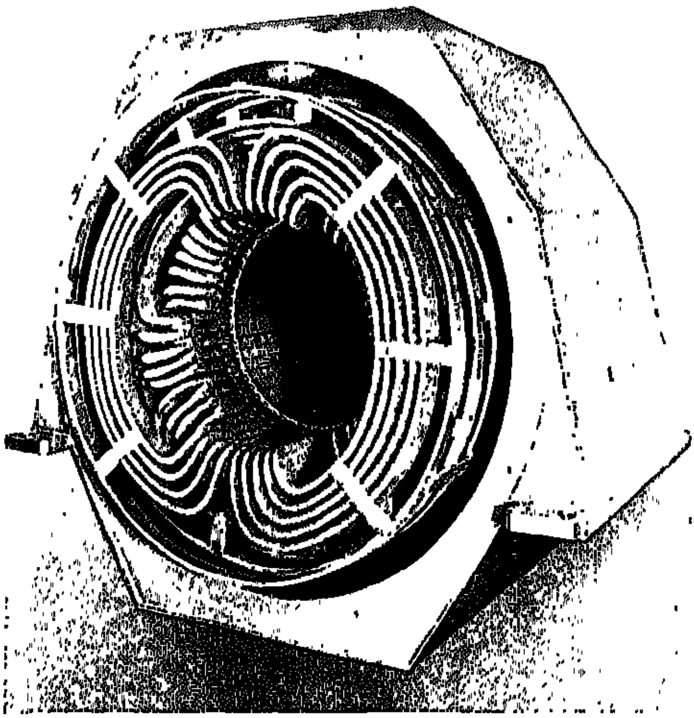


PLATE XI. THREE-PHASE, THREE-PLANE, SPLIT-PHASE HAIR-PIN WINDING

Courtesy English Electric Co. Ltd.

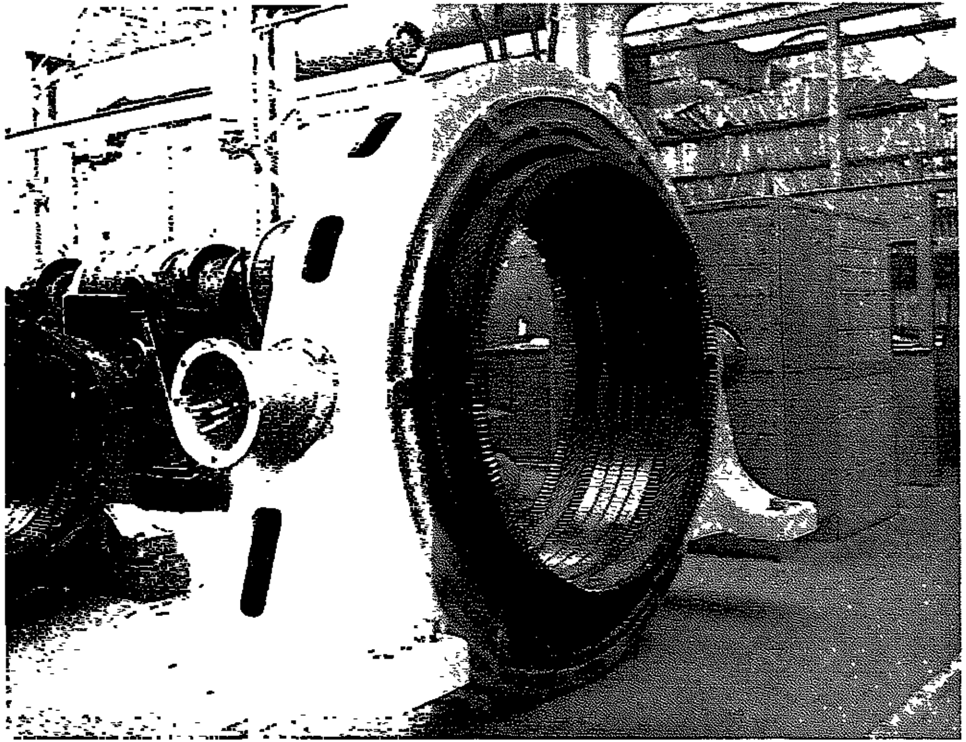


PLATE XII. A TWO-BAR-PER-SLOT, TWO-LAYER, HAIR-COIL, WAVE-WOUND STATOR WINDING

(Courtesy, English Electric Co. Ltd.)

phase shifts, according as the point P moves over the circle (*see* Fig. 8.14).

Let $OA = E_{2s}$ ($s = 1$), i.e. the voltage per rotor phase induced at standstill

AP = voltage in secondary of rotary transformer

AB = component of impressed voltage in phase opposition to E_2 and so speed reducing

PQ = voltage from secondary of current transformer B.

For the controlled range of speeds the wattless component of AP is much greater than is desirable, causing large magnetizing currents in the rotor. The demagnetizing component, introduced by the current transformer, viz. PQ, gives, with AP, the resultant voltage AQ, and the locus of Q is a circle, whose centre is displaced by the amount PQ from the circle followed by P.

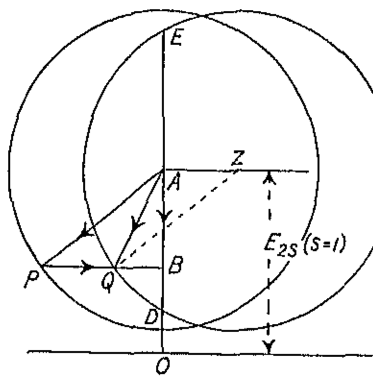


FIG. 8.14

The maximum speed corresponds to the point E, when Q falls on E, for then we have added to E_{2s} ($s = 1$) the voltage AQ. The minimum speed is obviously at D, when Q falls on D. AB corresponds to the induced armature voltage.

If ω_s = synchronous speed

s = slip

and ω = actual speed

$$\omega = \omega_s(1 - s) \quad . \quad . \quad . \quad . \quad (8.41)$$

Also
$$E_2 = sE_{2s} \quad . \quad . \quad . \quad (8.42)$$

$$\therefore \quad E_{2s} - E_2 = E_{2s}(1 - s) \quad . \quad . \quad . \quad (8.43)$$

and $\therefore$
$$\frac{E_{2s} - E_2}{E_{2s}} = 1 - s = \frac{\omega}{\omega_s} \quad . \quad . \quad . \quad (8.44)$$

$$\therefore \quad \omega = \omega_s \times \frac{E_{2s} - E_2}{E_{2s}} \quad . \quad . \quad (8.45)$$

and $E_{2s} - E_2 = OB$ (in Fig. 8.14).

The arc of the circle, between D and E in Fig. 8.14, represents the range of ω_{min} to ω_{max}.

$$\omega_{min} = \omega_s \frac{(E_{2s} - AD)}{E_{2s}} \quad . \quad . \quad . \quad (8.46)$$

and
$$\omega_{max} = \omega_s \frac{(E_{2s} + AE)}{E_{2s}} \quad . \quad . \quad . \quad (8.47)$$

and $E_{2s} = OA$ (in Fig. 8.14).

Without brush shifting, Dr. Schwarz, states that a speed range of 1 : 2 to 1 : 25 is obtainable in an economical manner.

The Two-stage Rotary Controller

We have two rotary potential transformers, primaries B and C and the stator windings of the motor connected to the supply lines. The secondaries of the two rotary potential transformers D and E, and the secondaries of the current transformers F, G, and H are connected in series with the armature of the motor. (*See* Fig. 8.15.)

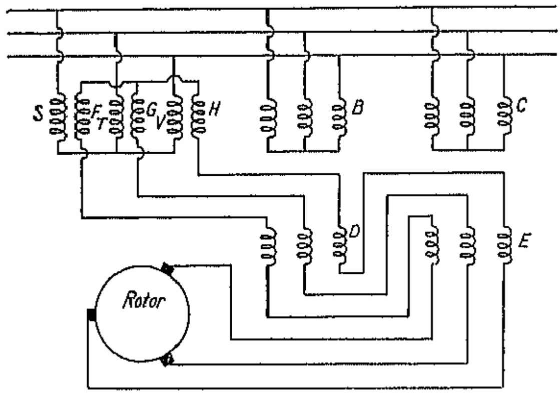


FIG. 8.15

In this case, the voltage AQ, which is the demagnetizing voltage, is constant over the whole range of speed. The idea, then, is to produce voltages in the rotary transformers, which have a speed controlling component, in phase, or in phase opposition with the induced rotor voltage, and also a demagnetizing component in quadrature with E_2.

Vector Diagram of Shunt Motor

Fig. 8.16, due to Blondel, shows the circle locus of the primary current, viz. the circle HBP, and the circle HKd, with centre Q the locus with brushes short-circuited.

OA = magnetizing current per phase and phase of the flux

OC = e.m.f. induced in the rotor per phase $= e_2$

$$= 1 \cdot 84 Z_2 \times \hat{\phi} \times s \times f \times 10^{-8}$$

where Z_2 = conductors in series per phase between two neighbouring brushes

$\hat{\phi}$ = maximum flux per pole, produced by the resultant of stator and rotor ampere-turns

$OE = V_2$ = applied voltage per phase to the rotor

$OD = E_{2R}$ = resultant voltage of OE and OC

I_2 = rotor current per phase lagging by an angle ϕ_2

AB = rotor current, referred to the stator

OB = stator current per phase

$OR = -e_1$ = component of applied volts to the stator to overcome the induced voltage, due to flux ϕ

$= \sqrt{2}\pi \times \hat{\phi} \times f \times T_s \times K_1 \times K_3 \times 10^{-8}$ volts

where T_s = turns in series per phase

K_1 = breadth factor

K_3 = coil span factor

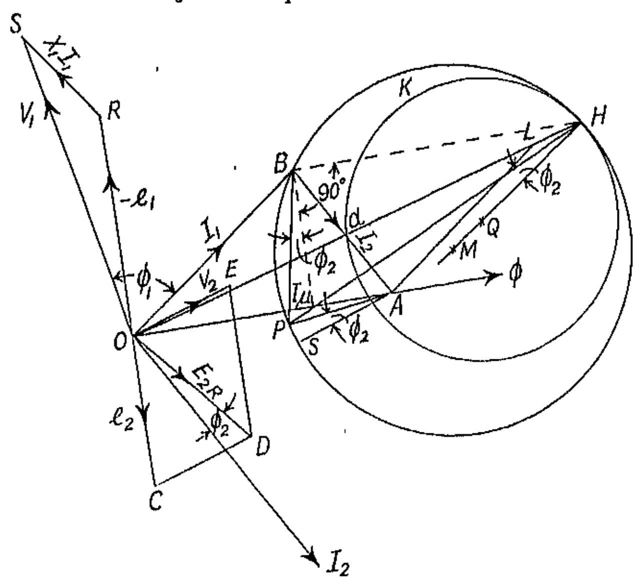


FIG. 8.16

The secondary current I_2 consists of two components, viz. BP and AP; BP is produced by e_2 and equals $\dfrac{e_2}{R_2 + jX_2}$

The r.m.s. value of this component

$$= \frac{1{\cdot}84 \times Z_2 \times \hat{\phi} \times s \times f \times 10^{-8}}{\sqrt{R_2^2 + X_2^2}}$$

The PA component is produced by V_2, the applied volts per phase, and $= \dfrac{V_2}{\sqrt{R_2^2 + X_2^2}}$

Assuming a constant secondary reactance, which is not strictly true, the point P is fixed for constant value of V_2 and brush-shift angle α. Draw BH and draw, also, OH making an angle of 90° with V_1 or OS. Then OH represents the ideal short-circuit current, and is proportional to the total leakage flux which balances the voltage V_1, assuming no resistance present. H is a fixed point since OH is fixed, and the angle $HBP = 90 + \phi_2$. Since P is a fixed point and H also fixed, PH is fixed, and it subtends an angle $90 + \phi_2$ at B; it follows the locus of the point B is the circle PBH, of which PH is a chord.

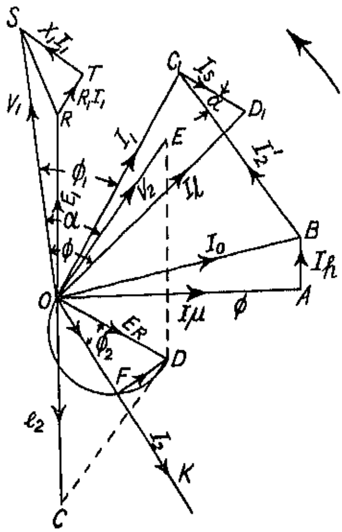


FIG. 8.17. CLEARER VECTOR DIAGRAM OF SHUNT MOTOR

Let M be the centre of this circle. Then the centre is found by bisecting PB and BH and drawing perpendiculars to intersect at M. The centre clearly lies on the line MH, which makes an angle ϕ_2 with PH. AL, parallel to I_1, represents the primary leakage flux and is proportional to I; OL is proportional to the total flux linking the stator. In our diagram there appears the angle SOE between V_1 and V_2. Actually the same source supplies both stator and rotor, the rotor being supplied from a transformer. To obtain the line current, we must add vectorially the current in the primary of the transformer feeding the rotor. If m is the ratio of the transformer, we must take a current $= \frac{I_2}{m}$ and add it vectorially, in such a way that it makes the same angle with both V_1 and V_2.

So all we need to do is to add $\frac{I_2}{m}$ at the angle α of brush shift, to I_1. The total line current, obtained by this vector addition of I_1 and $\frac{I_2}{m}$ at angle α with I_2', gives another circle, and over the motor range, the total current is smaller than the stator current, since the rotor returns energy to the supply circuit.

Fig. 8.17 gives a clearer vector diagram for the shunt motor. In this diagram—

OA = vector of magnetizing current and direction of flux

$OC = e_2$ = e.m.f. induced in rotor per phase at slip s by ϕ, lagging 90° behind ϕ

$OE = V_2$ = applied voltage per phase to the rotor

OD = resultant voltage per phase of the rotor = vector sum of OE and $OC = E_R$

OF = secondary current in rotor, viz. I_2, lagging by angle ϕ_2 behind OD, and

$$= \frac{E_R}{Z_2}$$

where Z_2 = impedance of the rotor per phase

$$\tan \phi_2 = \frac{X_2}{R_2}$$

where X_2 = leakage reactance of the rotor per phase

R_2 = resistance per phase of rotor

$OF = R_2 I_2$

$FD = X_2 I_2$

$$BC_1 = I_2' = -\frac{T_2 f_2}{T_1 f_1} \times I_2$$

where T_2 = turns in series per phase in the rotor

T_1 = turns in series per phase in stator

f_1 = winding factor of stator = $K_1 \times K_3$

f_2 = winding factor of rotor = $K_2 \times K_4$

AB = watt component of no-load current to overcome hysteresis loss

I_0 = no-load current per phase = OB

OC_1 = primary current of motor

$C_1D_1 = I_r = I_2 \times \frac{V_2}{V_1}$ = current in primary of transformer supplying the rotor

C_1D_1 is added at an angle of α with I_2', therefore—

OD_1 = vector of *line* current, including the current taken by the stator and that taken by the primary of the transformer, supplying the rotor = I_l

OR = component of impressed volts to overcome the back e.m.f. generated in the stator per phase, due to ϕ

$R_1 I_1$ = resistance drop in the stator per phase

$X_1 I_1$ = leakage reactance drop in the stator per phase

$OS = V_1$ = applied volts per phase to the stator

$\cos \angle C_1OS$ = power factor of the stator

$\cos \angle SOD_1 = \cos \phi$ = power factor of the motor

Input $= \sqrt{3}V_1I_1 \cos \phi$ (watts).

Electrical power delivered by the rotor to the line

$$= \sqrt{3}V_2I_2 \cos \angle EOK$$

Gross output of motor = input to stator — output of rotor (electrical) — I^2R loss in stator and rotor — iron loss in stator and rotor.

THE SCHRÄGE MOTOR

In this motor, e.m.f.s are injected into the secondary circuit of the motor, in accordance with the principle that speed control is effected when those e.m.f.s are either in phase, or in phase opposition, with

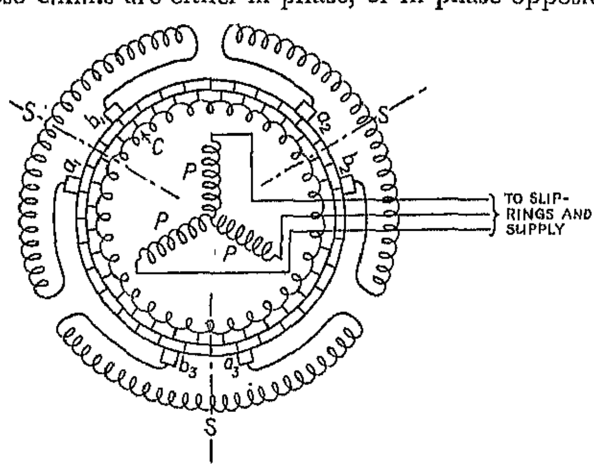


Fig. 8.18

the e.m.f.s in the rotor, induced by the revolving field. The primary of this machine is the rotor and the secondary the stator. The motor resembles the induction motor, but on the *rotor* are two sets of windings; one set, shown as a star-connected winding P in Fig. 8.18, is connected to three slip-rings, to which the a.c. voltage is brought. In the same rotor slots, is another winding; an ordinary d.c. lap winding, connected to a commutator and shown as winding C in our diagram. On the commutator there are three complete sets of brushes per pole pair, but each set is divided in two half-sets, and these half-sets are arranged to move in opposite directions on the commutator.

Thus, on the two-pole machine, there are six half-sets of brushes. Imagine the two half-sets to be in line on the commutator and connected to separate brush spindles, which are moved by a lever mechanism in opposite directions on the commutator. The stator has a three-phase winding and each phase is connected to the two halves of a brush set. Thus, one phase on the stator is connected to

half-brush sets a_1 and b_1, and similarly for the other phases. As before mentioned, these windings on the stator form the secondaries of the motor. Virtually it is an ordinary induction motor with an auxiliary winding on the rotor (i.e. the winding connected to the commutator).

A rotating field is produced by the three-phase currents, which rotate in the opposite direction to that of the rotor. E.m.f.s of slip frequency are generated in the secondary on the stator, as in any induction motor. Since the speed of the rotating field, relative to the rotor, is always the same, namely n_s, the frequency of the e.m.f.s induced in the auxiliary winding (i.e. the commutator winding) is independent of the speed of the rotor and always equal to line frequency. Therefore, the frequency of the e.m.f.s at the commutator brushes is always of slip frequency, and the frequency of the injected e.m.f. into the secondary is of slip frequency. This will be obvious if one remembers that for one definite speed, the distance of the brushes apart is fixed, and the brush position is fixed in space. Relatively to the brush half-sets, the relative speed of the field is the speed of the field—speed of rotor. Now the magnitude of the e.m.f. injected into the rotor depends on the distance apart of the half-brush sets. The greatest voltage is given when the brush sets a_1 and b_1, or a_2 and b_2, and a_3 and b_3 cover 180 electrical degrees on the commutator. When a_1 and b_1, and a_2 and b_2, and a_3 and b_3 are in line, i.e. a_1 in line with b_1, and a_2 in line with b_2, etc., the impressed voltage is zero, and the three stator windings are shorted. The machine in this case is an ordinary induction motor.

The speed control is continuous. The maximum speed range depends on the maximum voltage injected into the secondary winding from the auxiliary commutator winding, and this maximum voltage occurs when the half-brush sets (a_1 and b_1), (a_2 and b_2), (a_3 and b_3) embrace 180 electrical degrees on the commutator. For speed control only, the half-brush sets, when in line, must be at the centre of the phase of the secondary to which they are connected. In Fig. 8.18 the line bisecting (a_1 and b_1), (a_2 and b_2), and (a_3 and b_3) in each case lies at the centre of the stator phase considered. Thus, now we see why the brushes half-sets are moved in opposite directions simultaneously. It is to obtain voltages to inject into the secondary, and the magnitude of the voltages, thus injected, depends on the distances between a_1 and b_1, a_2 and b_2, a_3 and b_3. By insisting also on the condition that, when a_1 and b_1 are in line, a_2 and b_2 are also in line, and a_3 and b_3 in line, then the brush axis in each case must lie at the centre of the stator phase concerned, we ensure that the injected e.m.f. shall be in phase or in phase opposition with the induced e.m.f., due to the revolving field. Speed ranges of 3 to 1 and more may be obtained, but the output of this machine is limited by the voltage generated in the short-circuited coil by the revolving field. The e.m.f. of self-induction in the short-circuited coil is

proportional to the specific electric loading and depends on the range of speed required. The output per pole is, therefore, limited and, indeed, this motor is at a disadvantage in respect to the ordinary shunt motor, fed through the commutator, for, in this Schräge motor, the e.m.f. generated in the short-circuited coil is of full supply frequency, and is constant and independent of the speed. I refer, of course, to the voltage generated due to the revolving field.

Power-factor Correction

This necessitates that the injected e.m.f.s must lead the voltages generated in the secondary by the revolving field. For this purpose, the axis of a brush pair must be displaced from the axis of a stator phase by a given angle. If unity power factor is to be obtained, at all speeds, both brush supports must be moved by different amounts. About 10 per cent difference in distance between the brush supports is necessary. In the neighbourhood of synchronous speed, where the commutator voltage is small, the power factor is almost the same as the induction motor, but above synchronism, the power factor is improved by the negative leakage reactance of the secondary winding.

SPECIMEN CHARACTERISTICS: SCHRÄGE TYPE THREE-PHASE COMMUTATOR MOTOR

The characteristics of this motor, which is of the shunt type are given below in a test carried out on a small motor in the laboratory of Electrical Engineering of the University of British Columbia, Vancouver, Canada.

Test Machine. ASEA, Sweden. Type FS-6. No. 27598.

Output, kW: 2·2–6·6	Rating: continuous
Output, h.p.: 3–9	Primary connection: Y
R.p.m.: 600–1800	Cycles: 60 c/s
Primary volts: 220	Secondary volts: 33·5
Primary amperes: 17–24	Secondary amperes: 57

Observations—

(1*b*) No-load Test

Brush setting .	600	Start	1000	1200	1450	} at 60 c/s
R.p.m. . .	650	359	1036	1175	1480	

(1c) Load Test

	R.p.m.	Volts	Amperes	W_1	W_2	W_r	$\frac{W_2}{W_1}$	P.F.	Brake Load W lb	Brake Load S lb	Brake Load Net lb	B.h.p.	Efficiency (%)	f (c/s)
Setting "600"	470	228	24.8	5770	450	6220	0.08	0.56	18	3.25	18.8	4.20	50.0	59
	500	229	22.0	4570	220	4790	0.05	0.54	14	2.75	15.3	3.60	56.0	59
	530	229	19.2	3830	0	3830	—	0.50	10	2.60	11.4	2.90	56.5	60
	550	230	16.8	3230	− 260	2970	− 0.08	0.44	6	2.20	7.9	2.10	52.7	60
	590	231	15.1	2630	− 600	2030	− 0.23	0.34	2	2.00	4.0	1.10	40.5	60
	610	231	14.2	2400	− 750	1650	− 0.31	0.29	0	1.80	2.2	0.64	29.0	60
Setting "1000"	940	231	25.5	5810	2160	8070	0.37	0.78	16	3.20	16.8	7.42	69.0	60
	930	231	22.0	5000	1520	6520	0.30	0.73	12	2.80	13.2	5.77	66.0	60
	960	232	19.0	4170	870	5040	0.21	0.66	8	2.60	9.5	4.26	63.0	60
	960	233	16.4	3430	170	3600	0.05	0.54	4	2.10	5.9	2.66	55.0	60
	1000	233	14.6	2680	− 450	2230	− 0.17	0.38	0	1.80	2.2	1.07	36.0	60
Setting "1450"	1400	232	24.8	5720	3160	8880	0.55	0.89	12	2.90	13.1	8.60	72.5	60
	1410	232	20.6	4820	2050	6870	0.42	0.82	8	2.80	9.2	6.10	66.0	60
	1415	233	17.5	3910	1020	4930	0.26	0.70	4	2.30	5.7	3.80	57.5	60
	1425	233	15.2	3100	− 40	3060	− 0.13	0.41	0	1.90	2.1	1.40	34.5	60

Effective weight of brake arm = 4·0 lb.
Length of prony brake arm = 30 in.

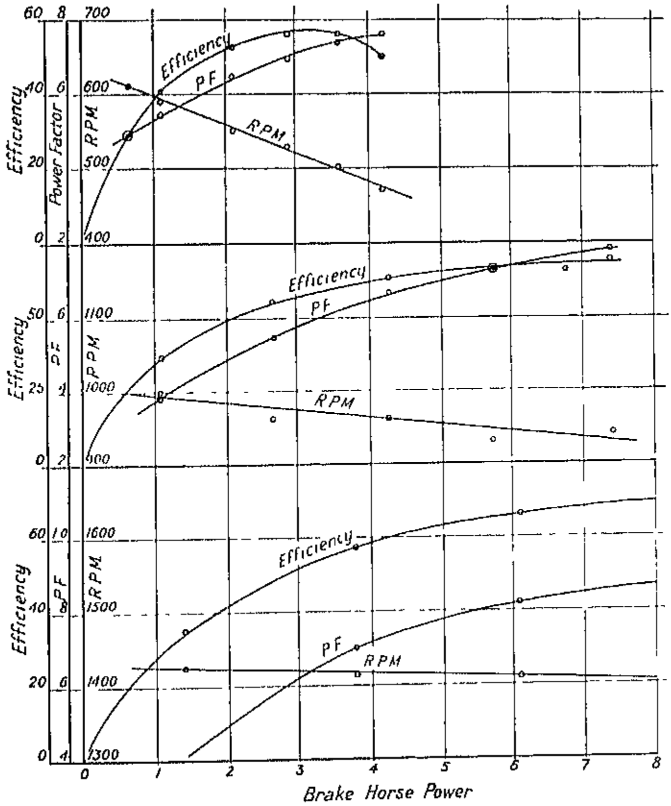


Fig. 8.19

(2*d*) Rotor at Standstill

Brush Setting	600			1000			1200			1450		
Primary volts .	233	233	233	233	—	—	233	—	—	233	—	—
Secondary volts	36	35·6	36	36	—	—	36	—	—	36	—	—
Injected volts .	15·4	15·0	15·8	3·0	4·75	2·7	1·0	0·5	0·5	10·8	9·75	11·0

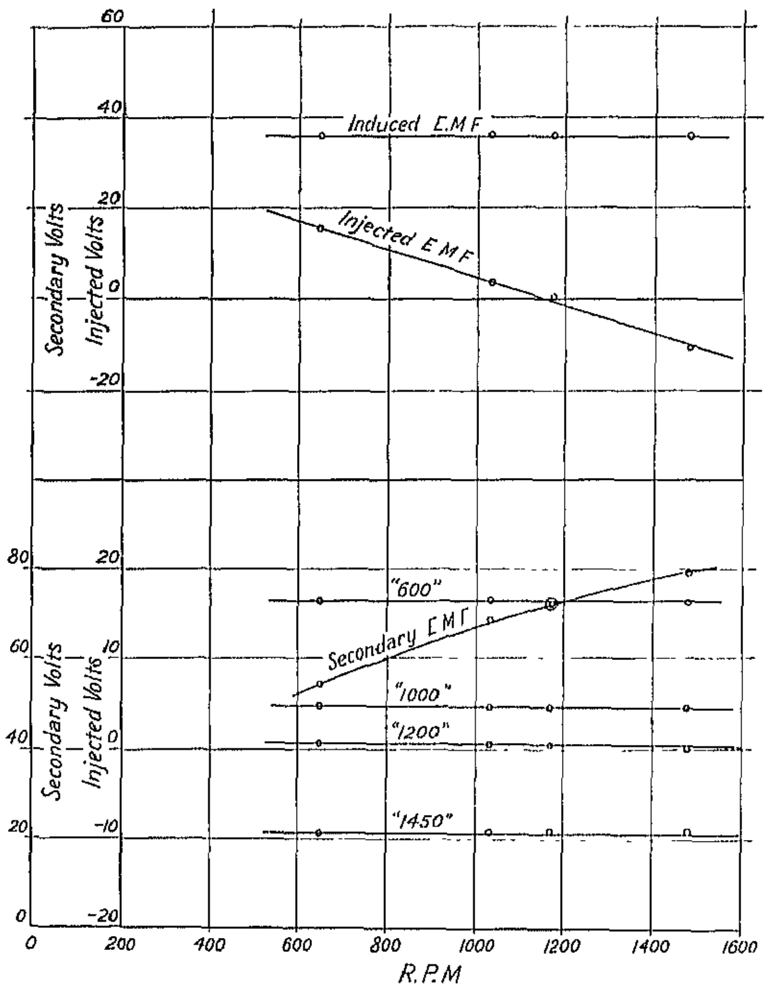


Fig. 8.20

(2*e*)

	R.p.m.	Brush Setting			
		600	1000	1200	1450
Primary volts	650	232	233	232	232
Injected volts		16·3	4·7	0·5	9·3
Secondary volts		54	54	54	54
Primary volts	1036	233	233	232	232
Injected volts		16·3	4·6	0·5	9·2
Secondary volts		68	69	68	68
Primary volts	1175	233	232	233	233
Injected volts		16	4·6	0·5	9·2
Secondary volts		71	72	72	72
Primary volts	1480	233	233	233	232
Injected volts		16·4	4·3	0	9·1
Secondary volts		79	79	79	79

Calculations

Power factors were determined from the ratio W_1/W_2.

$$\text{Net load on brake} = W - S + 4{\cdot}0$$

$$\text{Case (1) net load on brake} = 18 - 3{\cdot}25 + 4 = 18{\cdot}8 \text{ lb}$$

$$\text{Case (1) h.p.} = \frac{(2\pi)(470)(18{\cdot}8)}{33\,000} = 4{\cdot}2 \text{ h.p.}$$

$$\text{Case (1) efficiency} = \frac{\text{output}}{\text{input}} = \frac{(4{\cdot}2)(746)}{6220} = 50\%$$

CHAPTER IX

Phase Advancing

The power factor of lightly-loaded induction motors is very low, and likewise that of fully-loaded slow-speed motors. The deleterious effects of low power factor on the voltage regulation of the system are well known; and, further, the losses in generators and motors and transmission lines are increased due to low power factor. Further, in the design of induction motors, especially if of low speed, the output for a given frame is frequently reduced by the necessity of using shallow slots, for it is very important to reduce the leakage reactance. Several different types of machine have been devised for injecting a leading e.m.f. into the rotor circuits and thus to improve the power factor.

Leblanc's System

Leblanc proposed an arrangement of exciters for this purpose. Considering a two-phase induction motor, two exciters, one for each phase, are connected in series with the rotor winding, and the field of one exciter is excited by the rotor current of the second. These machines were ordinary commutating a.c. generators. With this arrangement an e.m.f. is injected into one phase, which is 90° out of phase with the current carried by that phase; and if the polarity of the poles is properly arranged, this is a leading e.m.f. The objection to this scheme was the excessive cost. Leblanc has devised an exciter which embodies, in one machine, all the phases, and this is of a very simple nature.

The armature is made like an ordinary drum-wound continuous-current armature, and is surrounded by a simple ring of laminations having inwardly projecting poles, but without field windings. If such an armature is provided with four brushes, placed 90° apart on the commutator, and connected to the four slip-rings of a two-phase rotor of an induction motor, and is run at a speed which is high compared with the frequency of the rotor circuits, it will have the effect of producing leading currents in the rotor circuit. The beauty of the exciter is that the armature currents excite the field and produce a flux in the armature which is in such a phase as to generate an e.m.f. in each circuit which is in quadrature with the current carried by the circuit. By proper design and the use of carbon brushes, the commutation can be made sufficiently good. The rotors of induction

motors of large power carry heavy currents, and this necessitates a commutator of considerable dimensions.

The Scherbius System

In the Scherbius phase advancer, the external ring of laminations is omitted, and the armature winding is embedded in slots, which are some distance from the periphery. The path for the magnetic field is, therefore, through the iron above the slots. When used in conjunction with a three-phase rotor, three sets of brushes are provided on the commutator at 120 electrical degrees apart. These brushes receive the rotor currents of slip frequency. The currents of slip frequency will produce a revolving field whose speed is

$$f/p \text{ revolutions per second}$$

where f = slip frequency

p = pairs of poles

Now suppose the rotor to be driven in the same direction as the field revolves. The speed of the field is independent of the speed of rotation of the armature. When the speed of the armature is equal to the speed of the rotating flux wave, no relative motion or cutting of the lines takes place. The self-induction effect disappears therefore. Above this speed, i.e. above the synchronous speed of the field, the reactance e.m.f. becomes negative; in other words, the current will lead.

Let ϕ = flux per pole in megalines

$2p$ = number of poles for which the armature is wound

f = slip frequency

n = number of revolutions per second at which the armature is driven

The star value of the e.m.f. injected for a wave-wound armature

$$e = \frac{p}{2} \times 0{\cdot}7 \times \phi \times \frac{Z}{100}(n - f/p)$$

where Z = the total number of active wires

The rating of the phase advancer is proportional to its volt-ampere capacity. The current is determined by the load, but the e.m.f. injected by the advancer is capable of adjustment. It is clear that if the motor is designed for a small natural slip, and the phase advancer has small ohmic and inductive losses, then the injected e.m.f. required becomes smaller, and the cost of the phase advancer will be less.

Fig. 9.1 shows the circle diagram for the motor with and without the phase advancer. It is taken from the article by the late Dr. Kapp.

In this figure, it is assumed that the ratio of turns in stator and rotor is 1 : 1. ED is proportional to the secondary or rotor current, and EM is proportional to the slip-ring voltage with the secondary open. The voltage drop in the rotor is $ER_0 = I_2R_2$, where $I_2 = ED$ and R_2 is the resistance of one phase of the rotor. Without the phase advancer the locus of D is the semicircle shown. With the phase advancer and at unity power factor, the point D moves to A, thereby reducing the

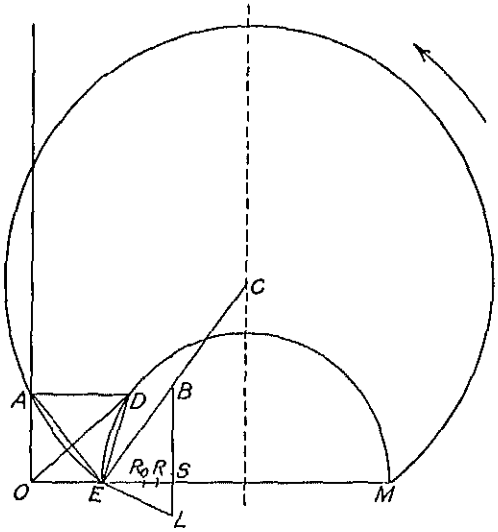


FIG. 9.1

stator current, but increasing the rotor current. The increased drop due to this increase in current is shown as R_0R. The resistance drop in the advancer is shown as R_0S. The slip voltage must have a power component ES. It must also have a wattless component equal to the e.m.f. introduced by the advancer minus the e.m.f. of self-induction lost in the advancer itself.

In the diagram, SL = e.m.f. of self-induction of advancer

and LB = e.m.f. of advancer

The wattless component of the slip voltage is SB.

The total slip voltage is EB, which is greater than ER_0 and, therefore, the slip is increased by using a phase advancer.

Dr. Kapp lays especial emphasis on the importance of a small natural slip. He also states that the cost of the set, capable of giving unity power factor under such conditions, need not be greater than the cost of a motor designed to work alone. This may be true, but it is open to question. There is no doubt whatever that the use of the advancer will enable one to get a greater output from the motor frame, and in any case the extra cost of the advancer is probably

justified. The locus of the primary current vector with phase advancer is the larger circle shown. In this it is assumed that the flux in the phase advancer is proportional to the rotor current, i.e. there is no saturation. If saturation sets in at the higher loads, injected e.m.f. and rotor current are no longer proportional, so that EB becomes less inclined and the centre C is lower. This enables compensation at, say, two-thirds of full load without over-compensating at overload.

Compensation is chiefly required at the lower values of the load, and so it may be advantageous to introduce a moderate degree of saturation in the advancer.

For small motors, the phase advancer may be mounted on the rotor shaft. It is then wound for a smaller number of poles than the main motor. In this case the speed n is fixed by the slip.

If n_s = synchronous speed of the set in revolutions per second

$$= \frac{f_1}{p_1}$$

where f_1 = supply frequency

p_1 = pairs of poles of induction motor

the speed of the induction motor

$$= (1 - s)\frac{f_1}{p_1} \qquad (9.1)$$

$$= (1 - s)n_s \qquad (9.2)$$

the speed of the field of the advancer

$$= \frac{sf_1}{p_2} \qquad (9.3)$$

$$= s\frac{f_1}{p_1} \times \frac{p_1}{p_2} = sn_s \times \frac{p_1}{p_2} \qquad (9.4)$$

therefore, the relative speed in this case

$$= (1 - s)n_s - sn_s\frac{p_1}{p_2} \qquad (9.5)$$

$$= n_s\left(1 - s - s\frac{p_1}{p_2}\right) \qquad (9.6)$$

and the star value of the injected e.m.f. for a wave-wound two-circuit advancer

$$= 0{\cdot}7\phi \times \frac{Z}{100} \times n_s\left(1 - s - s\frac{p_1}{p_2}\right) \times \frac{p_2}{2} \qquad (9.7)$$

where ϕ = flux in megalines

Z = total number of conductors on advancer

s = slip of main motor

p_1 = pairs of poles of main motor

p_2 = pairs of poles of the advancer

It will be seen that as the load decreases the slip decreases, and the term in brackets increases slightly with reduction of load. The injected e.m.f. increases slightly, therefore, at light load, in addition to the increase at light load compared with full load due to saturation effects. For large motors the phase advancer is driven from the main motor either by belt or by other means.

The Miles-Walker Advancer

Prof. Miles-Walker has designed a very ingenious type of advancer. In this machine two types of armature winding are used: in one, an

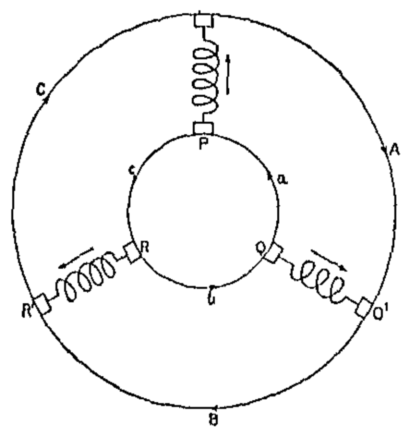


FIG. 9.2

open circuit star-winding with very wide brushes, suitable for low voltages and large currents; in the other, which is suitable for fairly high voltages and low current values, a closed-circuit winding with three sets of brushes, spaced 120 electrical degrees apart, is used. A diagrammatic scheme of connections is shown in Fig. 9.2. The inner circle represents the closed-circuit winding of the armature of the advancer, and the small letters a, b, and c show the three phases mesh-connected. Three brushes P, Q, and R bear on the commutator, and convey currents to the outer circle A, B, C, which represents the rotor windings of the induction motor, shown mesh-connected. The arrowheads show the direction along each conductor, which is taken as positive for the purpose of the clock diagram.

In series with P, Q, and R, the series exciting coils are connected. A suitable e.m.f. to inject into phase A is an e.m.f. in phase with $(a - b)$. The current in brush Q is $(b - a)$ and, therefore, if the poles under which the coils in phase A are passing are excited with $- Q$, the required condition is satisfied. The span of the armature coils is almost a pole-pitch, so that the coils in phase A will be passing under adjacent poles.

Since return paths have to be arranged and also compensating windings, it is necessary in order to make a simple mechanical arrangement of the coils to excite these poles with exciting conductors carrying currents $+ Q + Q - P_1 - R$.

Since $P + Q + R = o$, it follows that the above arrangement

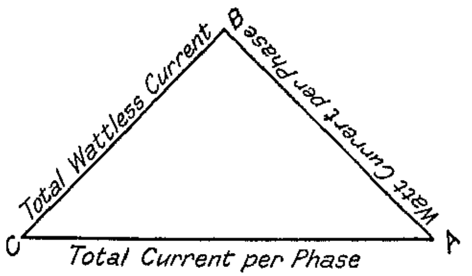


FIG. 9 3

will give $3Q$. The question whether $+ Q$ gives a forward or backward e.m.f. will depend on the direction of rotation and the hand of the winding.

To find the rating of the advancer, we must find the rotor standstill voltage per phase and the watt component of current in the rotor per phase. Knowing the power factor at full load, from the circle diagram we can determine the wattless current. If now the power factor is to be a leading one, we can find the amount of wattless current for this.

AB = watt current per phase
BC = total wattless current per phase
AC = total current of the phase advancer

The next thing to do is to determine the voltage of the advancer. From the value of the slip at full load we can determine the slip voltage at full load.

In Fig. 9.4, OE_a represents the slip voltage at full load, and Oa is set off at an angle from OE_a = angle CAB in Fig. 9.3. E_aR represents the resistance drop in the advancer and brushes, and RX the reactance drop in the advancer field coils. If, then, a voltage XY parallel to ba be added, a resultant voltage in phase with Oa is obtained.

This is the reason why the fields are excited with a current in phase with the sum of O_a and $- O_b$. The voltage to be generated is

scaled off from the diagram. As shown, the projection of OV on the vertical line gives OV_∞, which is greater than the slip voltage OE_a. If this voltage is greater than is required to drive the current through the rotor circuit, the slip will be reduced until the correct

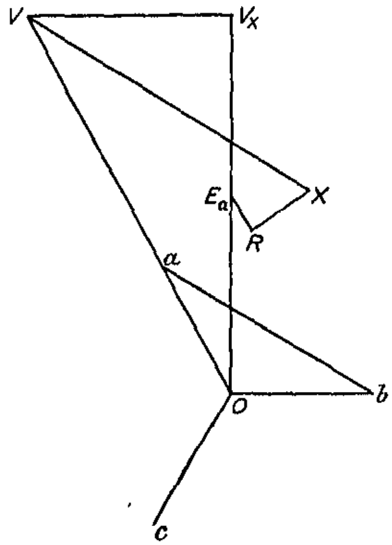


FIG. 9.4

working current for the load is reached. If OV_∞ is not sufficient, the slip is increased.

The Kapp Vibrator

The late Dr. Gisbert Kapp devised a type of phase advancer known as a vibrator. In this machine the principle is used that a conductor conveying an alternating current of low frequency, when placed in a magnetic field of constant strength, will vibrate and generate an e.m.f. leading the current by 90°.

The Kapp vibrator consists of a number of continuous-current armatures of the bi-polar type placed in bi-polar fields excited by direct current. The number of armatures required is equal to the number of phases in the rotor of the induction motor. Each armature is provided with a bi-polar closed-circuit winding and commutator, and each commutator has a pair of brushes. One brush of each commutator is connected to a slip ring of the induction motor, and the other brush is connected to a common neutral point when star connection is required.*

It is clear that the torque, acting on each armature, is proportional to the current, since the field is constant. The induced e.m.f.

* The armatures may be connected in delta for large rotor currents

in the conductor is proportional to the velocity of the conductor in the field.

We have torque $\propto$ current

acceleration $\propto$ torque $\propto$ current

e.m.f. $\propto$ velocity

If the current follows a sine law with respect to time, the acceleration will follow a sine law and the velocity will follow a cosine law, and, therefore, the e.m.f. will follow a cosine law.

The acceleration is zero when the velocity is a maximum, and, therefore, the e.m.f. generated is in quadrature with the current. We have yet to show that it is *leading* the current.

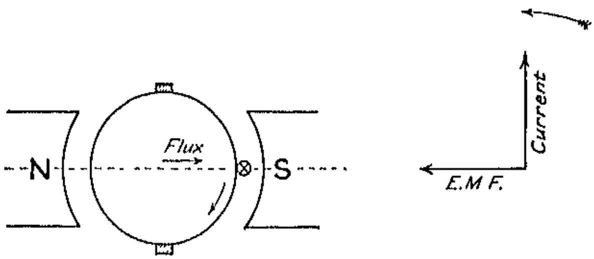


FIG. 9.5

In Fig. 9.5 let the field in which the armature vibrates have the direction from left to right, and let the direction of the current be considered positive when it flows away from the observer in the right-hand side of the armature. When the current is at its positive maximum, the torque is a maximum and the velocity zero, and the armature is momentarily at rest, but is on the point of moving in a clockwise direction. Now a clockwise movement will generate an e.m.f. in the right-hand half of the winding, which is towards the observer, i.e. which is negative. Thus, at this instant the e.m.f., which is zero, is about to assume a negative value. Hence, the vector diagram is as shown on the right, i.e. the e.m.f. leads the current by 90°. If Z = total number of conductors on one armature and ϕ = flux per pole in c.g.s. units, then the e.m.f. generated at any instant = e,

and $$e = \phi Z \frac{n}{60} p/a \times \frac{1}{10^8} \qquad (9.8)$$

$$= \phi Z \frac{\omega}{2\pi} \times \frac{1}{10^8} \text{ for two-pole armature} \qquad (9.9)$$

where n = revolutions per minute

p = poles

a = circuits in parallel

$p = a$ for two poles

$\omega = \frac{2\pi n}{60}$ = angular velocity in radians per second

If i = instantaneous value of the current, the instantaneous power

$$= ei = i\frac{\omega}{2\pi}Z\frac{\phi}{10^8} \quad . \quad . \quad . \quad (9.10)$$

If T = torque in metre-kilograms,

$$ei = 9{\cdot}81\,T\omega = i\frac{\omega}{2\pi}Z\frac{\phi}{10^8} \quad . \quad . \quad . \quad (9.11)$$

$$\therefore \qquad i = T \times \frac{9{\cdot}81 \times 2\pi \times 10^8}{Z \times \phi} \quad . \quad . \quad . \quad (9.12)$$

Also $$\omega = \frac{2\pi e \times 10^8}{Z\phi}$$

$$\int i dt = 9{\cdot}81 \times \frac{2\pi \times 10^8}{Z\phi} \times \int T\,dt \quad . \quad . \quad (9.13)$$

Let M = moment of inertia in kilogram-metres^2,

then $$\int i dt = \frac{9{\cdot}81 \times 2\pi \times 10^8}{Z\phi} \times \frac{M\omega}{9{\cdot}81} \quad . \quad . \quad . \quad . \quad (9.14)$$

$$= \frac{9{\cdot}81 \times 2\pi \times 10^8}{Z\phi} \times \frac{M}{9{\cdot}81} \times \frac{e \times 2\pi \times 10^8}{\phi Z} \quad . \quad (9.15)$$

$$= M \times \frac{4\pi^2 \times 10^{16}}{Z^2\phi^2} \times e \quad . \quad . \quad . \quad . \quad . \quad (9.16)$$

It is clear that the vibrator behaves like a capacitor whose capacitance

$$= \frac{M \times 4\pi^2 \times 10^{16}}{Z^2\phi^2} \text{ farads} \quad . \quad . \quad . \quad (9.17)$$

Also since $$e = \phi Z\frac{\omega}{2\pi} \times \frac{1}{10^8} \quad . \quad . \quad . \quad (9.18)$$

and $$9{\cdot}81\,M\frac{d\omega}{dt}\,\omega = ei \quad . \quad . \quad . \quad . \quad (9.19)$$

$$= \frac{i\omega}{2\pi}Z\frac{\phi}{10^8} \quad . \quad . \quad . \quad (9.20)$$

$$\therefore \qquad \frac{d\omega}{dt} = \frac{iZ\phi}{2\pi \times 10^8 \times 9{\cdot}81 \times M} \quad . \quad . \quad (9.21)$$

but $$i = \hat{I}\cos\omega_1 t \quad . \quad . \quad . \quad . \quad (9.22)$$

$$\therefore \qquad \int_0^{\pi/2} \frac{d\omega}{dt}\,dt = \frac{Z\phi}{2\pi \times 10^8 \times 9{\cdot}81 M} I \int_0^{\pi/2} \cos \omega_1 t\, dt \quad . \quad (9.23)$$

$$\omega_{max} = \frac{Z\phi}{2\pi \times 10^8 \times 9{\cdot}81 M\omega_1} I \quad . \quad . \quad (9.24)$$

therefore, maximum value of e.m.f. injected

$$= \frac{Z^2\phi^2}{4\pi^2 \times 10^8 \times 9{\cdot}81 M\omega_1} I \, . \quad . \quad . \quad (9.25)$$

$$\omega_1 = 2\pi \times \text{slip frequency} \quad . \quad . \quad . \quad (9.26)$$

$$\therefore \qquad \text{e.m.f. injected} \propto \frac{I}{\omega_1} \propto \frac{\text{current}}{\text{slip}} \quad . \quad . \quad (9.27)$$

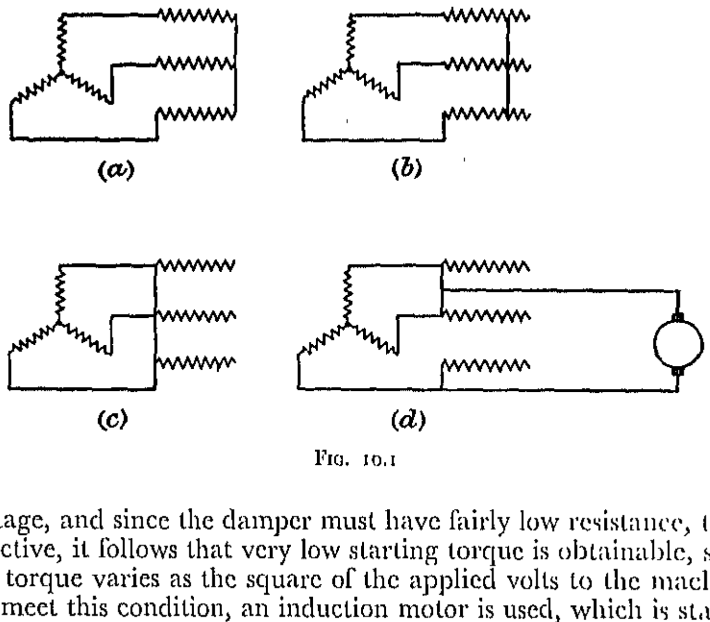
Since this ratio decreases only slightly as the load increases, the e.m.f. injected does not fall off proportionately with the load, but at a much lower rate, with the result that the effect of the advancer is relatively greater at low values of the load, and this is what is required.

It will be seen from the expression above for the injected e.m.f. how important it is to keep the moment of inertia of the armatures low. The armature must, therefore, be small in diameter and great in length; the air-gap must be small and the flux high, so that considerable saturation is required in teeth and core. Such a vibrator would be difficult to design for very large motors of slow speeds where the rotor currents are of necessity exceedingly high, but for machines of moderate output it has achieved very remarkable results.

CHAPTER X

Induction Motor with D.C. Secondary Excitation

FREQUENTLY, where constant speed is required and where it is desirable to have a large starting torque and high power factor under running conditions, the induction motor with d.c. excitation is used. The normal synchronous motor, with salient poles and usual damping winding, has a very low starting torque. To reduce the large, wattless, starting current, it is necessary to supply the motor with reduced



(a) (b) (c) (d)

FIG. 10.1

voltage, and since the damper must have fairly low resistance, to be effective, it follows that very low starting torque is obtainable, since the torque varies as the square of the applied volts to the machine. To meet this condition, an induction motor is used, which is started up in the usual way, by means of resistance in the rotor circuit. When the motor has reached full speed, the rotor is excited by a d.c. exciter, and, provided the slip is not too great, the machine drops into step and runs as a synchronous motor. The steps in the starting-up period are shown in Fig. 10.1.

With the connections shown in Fig. 10.1, one phase carries twice as much current as the other phases, and, therefore, the sectional

area of the conductors in that phase must be increased. It is possible, however, to use a rotor winding in which all the copper is evenly loaded. If one were to take an induction motor of reasonably good characteristics, and convert it into a synchronous motor, with constant field excitation, then it would be found that its characteristics, as a synchronous motor, are very poor, for the no-load current will be nearly equal to the full-load current, and the power factor and apparent efficiency are very low, except in a very narrow range just below the maximum output point or pull-out point.

On the other hand, by converting a slow-speed induction motor of large magnetizing current into a synchronous motor, then the characteristics as synchronous motor, with constant excitation, are very much better than those of the motor as induction motor. The reason for the unsatisfactory behaviour of a good induction motor, when converted to the synchronous type, is due to the large value of its synchronous impedance. In the induction motor, large magnetizing action of the stator current is desirable to produce the field, with as small a value of the current as possible. In the synchronous motor, large magnetizing action, on the part of the stator currents, calls for corresponding changes in d.c. excitation, and the higher the armature reaction, the more must the field excitation be changed with load, to maintain unity power factor. For good synchronous motor operation, low armature reaction is necessary. For good induction motor operation, the armature reaction must be high. For stable operation of the synchronous motor, a high synchronous reactance is necessary; but this high value is far lower than that in a good induction motor. A synchronous reactance of 50 to 100 per cent is high for a synchronous motor, but the synchronous reactance of a good induction motor may exceed 300 per cent.

Performance of Synchronous Induction Motor

Let $\boldsymbol{V}$ = impressed volts per phase, chosen as datum vector

$\boldsymbol{Z} = R + jX$ = synchronous impedance

X_1 = leakage reactance per phase

b_1 = exciting susceptance per phase

$$= \frac{\text{magnetizing current}}{\text{voltage induced by it}}$$

The effective reactance of armature reaction $= \frac{1}{b_1}$

Then
$$\boldsymbol{V} = \boldsymbol{E} + \boldsymbol{Z}\boldsymbol{I} \quad . \quad . \quad . \quad . \quad (10.1)$$

$\boldsymbol{E}$ = back e.m.f. or nominal induced voltage

$$\boldsymbol{I} = i_1 - ji_2 \quad . \quad . \quad . \quad . \quad (10.2)$$

$$\therefore \qquad \mathbf{V} = \mathbf{E} + (R + jX)(i_1 - ji_2) \qquad (10.3)$$

$$X = X_1 + \frac{1}{b_1}$$

$$\therefore \qquad \mathbf{E} = (V - Ri_1 - Xi_2) - j(Xi_1 - Ri_2) \qquad (10.4)$$

$$\therefore \qquad |E|^2 = (V - Ri_1 - Xi_2)^2 + (Xi_1 - Ri_2)^2 \qquad (10.5)$$

$$\therefore \qquad \mathbf{E} = e_1 - je_2 \qquad (10.6)$$

where
$$e_1 = V - Ri_1 - Xi_2 \qquad (10.7)$$

and
$$e_2 = Xi_1 - Ri_2 \qquad (10.8)$$

At unity power factor $i_2 = 0$,

and
$$E^2 = (V - Ri_1)^2 + X^2 i_1^2 \qquad (10.9)$$

At no-load and unity power factor $i_1 = 0$, $i_2 = 0$, and $E = V$.

The field excitation, required to keep $\cos\phi = 1$, at all loads, is given by equation (10.9).

From equation (10.9), we have

$$E^2 = V^2 - 2VRi_1 + R^2 i_1^2 + X^2 i_1^2 \qquad (10.10)$$

$$= V^2 - 2VRi_1 + Z^2 i_1^2$$

or
$$Z^2 i_1^2 - 2VRi_1 + V^2 - E^2 = 0 \qquad (10.11)$$

$$\therefore \qquad i_1 = \frac{2VR \pm \sqrt{4V^2R^2 - 4Z^2(V^2 - E^2)}}{2Z^2}$$

$$= \frac{VR \pm \sqrt{Z^2E^2 - X^2V^2}}{Z^2} \qquad (10.12)$$

The minimum value of

$$E = \frac{XV}{Z} \qquad (10.13)$$

For a given value of E, i.e. for constant excitation

$$i_2 = \frac{XV}{Z^2} \pm \sqrt{\frac{E^2}{Z^2} - \left(i_1 - \frac{RV}{Z^2}\right)^2} \qquad (10.14)$$

If Z is approximately equal to X, R negligible,

$$i_2 \doteqdot \frac{V}{X} \pm \sqrt{\frac{E^2}{X^2} - i_1^2} \qquad (10.15)$$

Under the same assumption, the maximum value of $i_1 = \frac{E}{X}$ for the radical becomes unreal for greater values of i_1.

The power input $= Vi_1 \qquad (10.16)$

The gross power output

$$= e_1 i_1 + e_2 i_2 \qquad (10.17)$$

$$= i_1(V - Ri_1 - Xi_2) + i_2(Xi_1 - Ri_2) \quad . \quad (10.18)$$

$$= Vi_1 - Ri_1^2 - Xi_1i_2 + Xi_1i_2 - Ri_2^2 \quad . \quad (10.19)$$

$$= Vi_1 - Ri^2 \quad . \quad (10.20)$$

where $i^2 = i_1^2 + i_2^2$

Therefore, equation (10.20) = input − copper losses.

The current in the field

$$= E \times b \quad . \quad (10.21)$$

The copper loss in the field

$$= E^2 \times b^2 \times r_f \quad . \quad (10.22)$$

where r_f = field resistance

The iron loss $= E^2g \simeq V^2g \quad . \quad (10.23)$

where g = conductance

Therefore, the net mechanical output

$$= Vi_1 - i^2R - E^2b^2r_f - E^2g - (\text{friction} + \text{windage}) \quad . \quad (10.24)$$

From these equations, it is simple to calculate the performance as a synchronous motor.

These machines are, usually, designed to give an overload capacity of 70 per cent for $\cos\phi = 1$, or 90 per cent for a leading power factor of 0·9. Should the machine fall out of step as a synchronous motor, it will continue to run as an induction motor, since the maximum overload capacity as induction motor is greater than that as synchronous motor, and it will automatically re-synchronize when the peak load is passed.

Further Analysis

We will now enquire into the phenomena which arise during the starting period, and how the design of the motor affects such phenomena. Consider first the expressions for the torque in the machine as induction motor and also as synchronous motor. We have seen that, in the induction motor, the torque in synchronous watts

$$\frac{m_2 s E_2^2 R_2}{R_2^2 + s^2 L_2^2 \omega^2} \quad . \quad (10.25)$$

where m_2 number of phases in the rotor

R_2 rotor resistance per phase

E_2 volts per rotor phase at standstill

L_2 coefficient of self-inductance of each rotor phase in henrys due to leakage flux

s = slip

$\omega = 2\pi \times f$

where f = frequency

Maximum torque occurs at a slip

$$s_m = \frac{R_2}{L_2\omega} \quad . \quad . \quad . \quad . \quad (10.26)$$

$$s_m = \frac{\text{rotor resistance per phase}}{\text{rotor reactance per phase at standstill}}$$

Therefore, maximum torque per phase

$$= \frac{E_2^2}{2L_2\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.27)$$

$$= \frac{\text{square of rotor volts per phase at standstill}}{2 \times \text{rotor reactance per phase at standstill}}$$

This is obtained, of course, by substituting for s the value $\frac{R_2}{L_2\omega}$ in equation (10.25).

The maximum torque per phase, viz. $\frac{E_2^2}{2L_2\omega}$, can also be written as

$$\frac{E_2}{2} \times \frac{E_2}{L_2\omega} = \frac{E_2}{2} \times I_{scr} \quad . \quad . \quad . \quad (10.28)$$

where I_{scr} = *ideal rotor short-circuit amperes per phase.*

Now assume this induction motor, with wound rotor, is operated as an auto-synchronous motor. As synchronous motor, the gross mechanical power developed per phase

$$= \frac{E_2[E_1 \cos(\gamma - \theta) - E_2 \cos \gamma]}{\sqrt{R_1^2 + L_1^2\omega^2}} \quad . \quad . \quad (10.29)$$

where E_2 = generated e.m.f. per phase in stator (r.m.s. value)

E_1 = applied volts per phase in stator (r.m.s. value)

R_1 = resistance, in ohms, per phase of the stator

$L_1\omega$ = *synchronous* reactance per phase, in ohms, of the stator

$$\tan \gamma = \frac{L_1\omega}{R_1} \quad . \quad . \quad . \quad . \quad (10.30)$$

Now, generally, $\lambda \simeq 90°$, i.e. the synchronous reactance is very large compared with the resistance.

In this case, the gross mechanical power developed per phase

$$\simeq \frac{E_2E_1 \sin \theta}{\sqrt{R_1^2 + L_1^2\omega^2}} \quad . \quad . \quad . \quad . \quad (10.31)$$

$$\simeq \frac{E_2E_1 \sin \theta}{L_1\omega} \simeq E_2I_{sc} \sin \theta \quad . \quad . \quad (10.32)$$

The angle θ is, of course, the angle between the applied volts per phase and the component of applied volts, which is equal and

opposite to the back e.m.f.; $180 - \theta$ is the angle between the applied volts and back e.m.f.

The maximum, gross, mechanical power per phase as synchronous motor (assuming $\lambda = 90°$)

$$= E_2 I_{s0} \text{ watts}$$

The maximum torque, per phase, will also be proportional to this expression, and is equal to it when expressed in synchronous watts per phase. Now let us assume that this machine is operating as a synchronous motor, and let the direct current in one phase $= \sqrt{2}I_r$, and $\frac{I_r}{\sqrt{2}}$ in each of the other phases, where I_r = full-load rotor current as induction motor. The copper losses in the two cases are then equal, i.e. when the machine is working as induction motor, or as synchronous motor. The magnetizing current, as induction motor, to produce normal flux, as a fraction of full-load current, will depend on the number of poles for which the motor is wound and will vary from 25 per cent on high speed motors to 60 or 70 per cent or more on very slow speed machines. Let us assume that it is one-third of full-load current.

The corresponding d.c. excitation for normal flux will be $\sqrt{2} \times \frac{1}{3}I_r = 0{\cdot}472I_r$. This leaves $0{\cdot}942I_r$ for producing leading current, and the r.m.s. value of such leading current will be $0{\cdot}666I_r$. Assuming the watt component of the current remains unaltered, the power factor will be 0·830 leading. The total stator current will not be altered greatly, from the value it had as induction motor. Assuming a leakage reactance per stator phase of 10 per cent, the value of E_2 will be roughly $1{\cdot}07\ E_1$.

It is clear that such a machine, thus excited, could be operated as a synchronous motor and work in the neighbourhood of 0·8 leading power factor, with the same rotor heating in each case. Now what about the maximum mechanical power exerted in this case? The ideal short-circuit current in the stator will be but equal to the full-load current, and E_2 (approximately) $= 1{\cdot}07E_1$.

Therefore, maximum mechanical power as synchronous motor, or maximum torque in synchronous watts $= 1{\cdot}07E_1 \times I_1$ per phase. Now, as induction motor, this same machine will have a maximum torque, in synchronous watts per phase

$$= \frac{E_2}{2} \times I_{sor}$$

$$= \frac{E_1}{2} \times I_{sos} \quad . \quad . \quad . \quad . \quad (10.33)$$

where I_{sos} = ideal short-circuit current in the stator per phase.

Now I_{sos} may be roughly five times full-load current (this depends

on the number of poles). Therefore, maximum torque in synchronous watts, *per phase*

$$= 2{\cdot}5E_1I_1$$

where I_1 = full-load stator current

It is clear, therefore, that such an induction motor, when working as synchronous motor, has much too small a synchronizing power and also too little overload torque. In other words, it would pull out of step with small overload. Let us examine the conditions under which this machine will come into step, during the period of synchronizing. Let us assume that the machine is started up, under load, as an induction motor, and that it is running steadily, as such, at a slip s. The corresponding torque is, say, T_M, which is assumed constant. The relation between slip and torque is a linear one in the region from slip s to zero slip. The machine has to be accelerated from slip s to zero slip, during synchronizing. The synchronizing torque must have sufficient value to overcome the constant load torque, and also be sufficiently great to accelerate the rotor from slip s to zero slip.

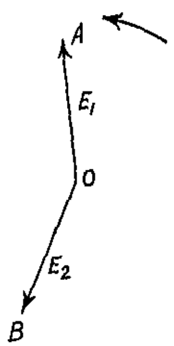


FIG. 10.2

Let OA (Fig. 10.2) represent the vector of applied volts per phase and OB represent the vector of generated e.m.f. per phase due to the field excitation. OA rotates at a speed corresponding to the supply frequency, whereas OB rotates at a speed corresponding to frequency of rotation, or the frequency of generated e.m.f.

The relative angular velocity is proportional to the slip, and the time of one revolution of relative velocity is the time of one slip period, i.e. E_1 is assumed stationary and E_2 revolves at slip frequency. When the vector E_2 is in the first and fourth quadrants, we have generator action, and when E_2 is in the second and third quadrants, motor action. Whether the machine will pull into step, or not, will depend on whether, or not, the synchronizing torque action, during the half-cycle of motor action is sufficient, or not, to accelerate the machine to full speed. It will be noted that the average value of the synchronizing torque, at any speed, other than synchronous speed, over any number of slip periods, is zero.

Let s = slip

f = supply frequency

The slip frequency $= sf$ and the time of one slip period $\frac{1}{sf}$

The synchronizing torque $\propto E_1E_2 \sin\theta$, say, $\hat{T}_s \sin\theta$.

Let M = moment of inertia of the rotor

T_M = torque of the induction motor at slip s

T_l = torque of the load, assumed constant
s_l = slip corresponding to load torque

Then we have the equation of motion,

$$T_M + \hat{T}_s \sin\theta = T_l + M\frac{d\omega}{dt} \quad . \quad . \quad (10.34)$$

Now
$$\omega = \omega_s(1 - s)$$

$$\frac{d\omega}{dt} = -\omega_s \frac{ds}{dt} \quad . \quad . \quad . \quad (10.35)$$

Also
$$\frac{T_M}{T_l} = \frac{s}{s_l} \quad . \quad . \quad . \quad . \quad (10.36)$$

$\therefore$
$$T_M = T_l \frac{s}{s_l} \quad . \quad . \quad . \quad . \quad (10.37)$$

Now the relative angular velocity of the rotor, with respect to the revolving field of the stator

$$= s\omega_s = \frac{d\theta}{dt \times p/2}$$

here θ is an electrical angle, and we must divided θ by the number of pairs of poles to get the mechanical angle.

$\therefore$
$$M\frac{d\omega}{dt} = -M\omega_s \frac{ds}{dt} \quad . \quad . \quad . \quad (10.38)$$

but
$$s\omega_s = \frac{d\theta}{dt \times p/2} \quad . \quad . \quad . \quad (10.39)$$

$\therefore$
$$M\frac{d\omega}{dt} = -\frac{M}{p/2}\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad (10.40)$$

Also
$$T_M = \frac{T_l}{s_l \cdot \omega_s} \times \frac{d\theta}{dt} \times \frac{1}{p/2} \quad . \quad . \quad (10.41)$$

$\therefore$
$$\frac{T_l}{s_l \times \omega_s \cdot p/2}\frac{d\theta}{dt} + \hat{T}_s \sin\theta + \frac{M}{p/2}\frac{d^2\theta}{dt^2} = T_l \quad . \quad (10.42)$$

or
$$A\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + T_s \sin\theta - T_l = 0 \quad . \quad . \quad (10.43)$$

where A and B are the coefficients of $\frac{d^2\theta}{dt^2}$ and $\frac{d\theta}{dt}$ in equation (10.42).

The solution of equation (10.42) involves the use of elliptic functions. It is similar to the equation for the oscillation of a pendulum with large amplitude of swing.

An approximate solution can be obtained by a process of successive approximations, by using the expansion for sin θ, viz.

$$\sin \theta = \theta - \frac{\theta^3}{\angle 3} + \frac{\theta^5}{\angle 5}$$

The equation is first solved by using the first term θ for sin θ, and the value of θ, obtained from this solution, is substituted in the differential equation with the second term added, viz. $\sin \theta = \theta - \frac{\theta^3}{\angle 3}$, and a second solution of the differential equation is then obtained. This process is repeated. This step-by-step method is long and laborious.

We may get some idea as to whether the machine will come into step for any given value of the slip in the following manner. Assume the machine is started up as an induction motor under a constant load torque T_L. The slip will be known, for this torque, for any motor.

Let T_{S_a} = average value of the synchronizing torque during one-quarter of a slip period

T_{M_a} = induction motor torque, average value during the speed increase from slip s to slip 0

M = moment of inertia of revolving parts

Then, assuming average values of T_S and T_M during the accelerating period from speed $\omega_0(1 - s)$ to ω_0, we have

$$T_{S_a} + T_{M_a} - T_L = M\frac{d\omega}{dt} \quad . \quad . \quad (10.44)$$

$$\int_{\omega_0(1-s)}^{\omega_0} d\omega = \frac{T_{S_a} + T_{M_a} - T_L}{M}\int_0^{t/4} dt \quad . \quad . \quad (10.45)$$

$$t = \text{time of one slip period} = \frac{1}{sf} \quad . \quad . \quad (10.46)$$

$$\therefore \qquad s\omega_0 = \frac{T_{S_a} + T_{M_a} - T_L}{M} \times \frac{t}{4} \quad . \quad . \quad (10.47)$$

$$= \frac{T_{S_a} + T_{M_a} - T_L}{M} \times \frac{1}{4sf} \quad . \quad . \quad (10.48)$$

from which

$$s = \sqrt{\frac{T_{S_a} + T_{M_a} - T_L}{4Mf\omega_0}} \quad . \quad . \quad . \quad (10.49)$$

This is the limiting value of the slip, corresponding to an average synchronizing torque, and an average induction motor torque, during the one-quarter of a slip period in which the speed rises from $\omega_0(1 - s)$ to ω_0. The machine is assumed to come into step without oscillation. The synchronizing power per phase $= E_2 I_{s0} \sin \theta$ and

the average value of sin θ, over a quarter of a slip period (for constant E_2 and I_{s0})

$$= \frac{4}{\pi}\int_0^{\pi/4} \sin \theta d\theta = \frac{4}{\pi}\Big[-\cos \theta\Big]_0^{\pi/4}$$

$$= \frac{4}{\pi}\left[-\frac{1}{\sqrt{2}} + 1\right] = 0{\cdot}373 \quad . \quad . \quad . \ (10.50)$$

Therefore, the average value of the synchronizing power per phase, during one-quarter of a slip period $= 0{\cdot}373E_2I_{s0}$, and taking the case of $E_2 = 1{\cdot}07E_1$ and $I_{s0} = I_1$.

The average value of the above *per phase* $= 0{\cdot}4E_1I_1$.

$$\frac{T_{S_a}\omega_0}{550} \times 746 = 0{\cdot}4E_1I_1 \times m_1$$

ω_0 = angular velocity of rotor in radians per second

$$\therefore \qquad T_S = \frac{550 \times 0{\cdot}4 \times E_1I_1m_1}{\omega_0 \times 746}$$

$$= \frac{0{\cdot}295E_1I_1m_1}{\omega_0} \text{ lb-ft} \quad . \quad . \quad . \ (10.51)$$

The limiting value of the slip to come into step is given by equation (10.49), when this value of T_S is substituted and also the average induction motor torque at a slip $= \frac{s}{2}$

One fact emerges very clearly, and that is that the induction motor, used as auto-synchronous motor has very low synchronizing power, and, indeed, will not pull into step against any appreciable resisting torque.

The question arises then as to how the design of the motor should be modified to make it suitable for the purpose of running as an auto-synchronous machine. A glance at the equation for the gross mechanical power developed, and the corresponding torque will show that it is necessary to increase the value of I_{s0}. This can be done, of course, by increasing the ratio of field ampere-turns to armature ampere-turns per pole, and this involves an increase in the air-gap length.

It would be desirable to have a maximum torque, or pull-out torque, as a synchronous motor equal to twice the full-load torque. The maximum power, developed by the synchronous motor, per phase $= \frac{E_1^2}{4R}$, but this value is obtained only when $\theta = \gamma$, and $E_2 = \frac{E_1}{2 \cos \gamma}$, i.e. for ranges of excitation outside practical limits.

The power developed *per phase* $= E_2 I_{s0} \sin \theta$ and the maximum value of this, when $\lambda = 90°$

$$= E_2 I_{s0} \quad . \quad . \quad . \quad . \quad (10.52)$$

The average flux density, in the teeth of induction motors, is probably about 12 000 to 13 000 lines per cm², and so it will not be possible to increase this by more than 30 per cent without introducing excessive saturation. Assume $E_2 = 1{\cdot}3E_1$, then it will be necessary to make I_{s0} equal to $1{\cdot}535 I_1$ to obtain a maximum power per phase equal to twice the normal input per phase at unity power factor. This means an increase of 53·5 per cent in the value of the r.m.s. current in the rotor, or a corresponding d.c. current of 2·17 times the r.m.s. value of the normal induction motor current in one phase. In the other two phases, which will be connected in parallel, the d.c. current will be 1·085 times the normal r.m.s. rotor current. More room will be needed, in the rotor slots, i.e. deeper slots must be used, which will necessitate a larger rotor diameter, than would be required in an ordinary induction motor, for the same output. It might be thought desirable to increase the number of rotor turns per coil, and so keep down the current for exciting purposes, but the voltage generated between the slip-rings at the time of start, limits this increase. The torque in synchronous watts, as an induction motor

$$= m_2 E_r I_r \cos \phi_r$$

where E_r = volts per rotor phase at standstill

$$\therefore \qquad m_2 E_r I_r \cos \phi_r = \frac{\text{b.h.p.} \times 746}{(1 - s)} \quad . \quad . \quad (10.53)$$

$$\text{i.e.} \qquad E_r I_r = \frac{\text{b.h.p.} \times 746}{m_2 (1 - s) \cos \phi_r} \quad . \quad . \quad (10.54)$$

Taking a value of $s = 0{\cdot}025$ at full load

and $\cos \phi_r = 0{\cdot}95$

and $m_2 = 3$

we have $E_r I_r = 268$ b.h.p. . . . (10.55)

The r.m.s. value of induction motor, rotor, current

$$= \frac{268 \times \text{b.h.p.}}{E_r} \quad . \quad . \quad . \quad (10.56)$$

The corresponding direct current, for excitation, at unity power factor for the overload capacity of twice

$$= \frac{2{\cdot}17 \times 268 \times \text{b.h.p.}}{E_r}$$

$$= \frac{582 \times \text{b.h.p.}}{E_r} \quad . \quad . \quad . \quad . \quad (10.57)$$

and in the other two phases

$$= \frac{296 \text{ b.h.p.}}{E_r} \quad . \quad . \quad . \quad . \quad (10.58)$$

The power required for excitation in watts (approximately)

$$= \frac{sE_r \times 582 \times \text{b.h.p.}}{E_r} \times 1{\cdot}5$$

$$= s \times 870 \times \text{b.h.p.}$$

$$= s \times 0{\cdot}87 \times \text{b.h.p. (in kW)} \quad . \quad . \quad (10.59)$$

where s = slip

The I^2R loss in the rotor, for the overload capacity stated, will be 2·36 times that in the rotor, when running under full-load current, as an induction motor. Clearly the rotor slots will need to be much deeper to accommodate the necessary copper. This will mean increasing the diameter of the rotor, except perhaps on large motors. At unity power factor in the stator, considerable reduction in the stator copper losses will result, over those in the induction motor.

Pulling Into Step

The most favourable position of the rotor, with respect to the application of the d.c. excitation is that in which the synchronous torque is

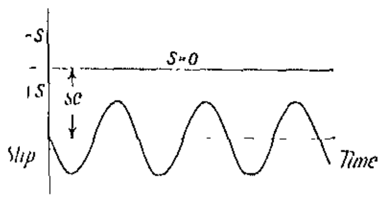


FIG. 10·3

positive, i.e. when the synchronous torque reduces the slip. Should the d.c. excitation be applied at an instant of time when the synchronous torque is negative, the slip is increased, and if the synchronous torque is not sufficient, during the positive half-wave to reduce the slip from its increased value to zero, the machine will not synchronize.

Fig. 10.3 shows variation of slip, when the d.c. excitation is applied, when θ = 180. Fig. 10.4 shows good position for rotor, when d.c. excitation is supplied.

We may plot the various torques graphically as ordinates and θ as abscissae.

The induction motor torque

$$T_M = T_e \times \frac{s}{s_t}$$